A Universe on a Lattice

The physics of cellular automata
— how much variety can a simple rule generate?

Edition: 13.0 (English, reordered for study, with a subtitle on every chapter)
Date: 2026-11-06 (thirteenth edition)
Basis: forty-one numerical experiments (first generation), twelve new calculations for the second edition, thirty-one calibrations for Part XII, a further one hundred and twelve for Part XIII and the Chapter 6 supplement, eight for Chapter 33 and Part XVIII, thirteen for Chapter 69, fifteen for Chapter 70, twelve for Chapter 71, thirteen for Chapter 72, and **two hundred and seventeen for Parts XIX-XXIV (Chapters 74-83), numbered K198-K399**, plus **twenty for Part XXVIII (Chapter 86), numbered K408-K427**, plus **eight checks for Part XXIX (Chapter 87), numbered K428-K435, without prior declaration** (all marked 🆕)

Figures. All figures in this edition are labelled in English. Part XII adds ten new figures; Part XIII and the Chapter 6 supplement add eleven; Chapter 33 and Part XVIII add two more; Chapter 69 adds one figure and one animation; Chapter 70 adds one more figure; Chapter 71 adds one more; Chapter 72 adds one more still; **Parts XIX-XXIII add twenty, and Chapters 1 and 2 gain two explanatory figures on cellular automata themselves - in Japanese and English alike**. Chapter 86 adds one more, also in both languages.

Preface — the wish for a simple law

P-1. Why this book

It is natural to wish that the world runs on a simple law. Conway's Game of Life was the first toy that gave the wish a concrete shape. Three lines of rule, and out come gliders, self-reproduction, and universal computation.

The rule is short. The behaviour is endlessly rich.

Then perhaps the universe is like that too. This book exists to take that question and **pin it down with calculation rather than leave it as a wish**.

The outcome, stated first. **The wish is granted almost everywhere.** Causality and hard-core statistics follow **automatically** from a simple local rule. Interactions and the exponential hierarchy appear in models with local constraints and an electric term (a restriction made in the thirteenth edition, §87-6). The uncertainty relation and $E = mc^2$ follow automatically once the quantum algebra of Chapter 2 is in place. On the side of Einstein's equations, what could be measured are **necessary conditions** such as the first law (the linear-order claim was retracted in the thirteenth edition, the second-order claim in the eleventh; §85-2, §87-2).

What was not granted comes down to four things. And all four reduce to the **same two properties**: that the lattice is **regular**, and that the algebra of observables is **commutative**.

Discreteness itself was never the obstacle. Regularity and commutativity were.

This is good news for the wish. Simplicity does not have to go. What has to go is something that had been quietly bundled together with it.

P-2. On the order of this book

This book is not written in the order of the investigation. The forty-one first-generation experiments doubled back repeatedly and contain four retractions. Laying them out chronologically teaches nothing.

So the material has been rearranged into a **study order**.

Chapter 0 :	what a cellular automaton is (no prerequisites, all figures)
Part I :	make precise what has been assumed
Part II :	count what comes for free (where the wish is granted)
Part III :	the first wall, and how to get past it
Parts IV--IX :	time, quantumness, fields, spacetime, matter, gravity
Part X :	summarise the gap; list what is required
Part XI :	evaluate candidate theories for the real world
Part XII :	take one candidate all the way to an experiment
Parts XIII--XVIII :	field equations, horizons, rotation curves, simultaneity, correlation, and a summary
Parts XIX--XXI :	the problem of time, measure and arrow, and the trilemma
Part XXII :	the deformation algebra of irregular causal sets
Parts XXIII--XXIV :	rotation curves again, and what changed since the ninth edition
Part XXV :	differences and absolutes — Casimir, Jacobson, the hydrodynamic limit
Part XXVI :	why gravity is weak — dimension fixes the exponent, coupling the coefficient
Part XXVII :	external verification of the tenth edition — six corrections
Part XXVIII :	The second law — does a quantum theorem fit on the lattice?
Part XXIX :	Peer review of the twelfth edition — two retractions, and assertions cut down
Chapter 88 :	closing

But **the derivations have not been cut**. Compressing to conclusions robs the reader of the experience of having found something out. In Parts III and VIII in particular, failed hypotheses and failed calibrations are kept in the main text. Appendix C records every error.

P-3. Notation

Mark	Meaning
✅	Theorem, or numerically established
⚠️	Partial result, conditional
❌	Refuted, or an obstacle of principle
🆕	Newly computed for this book
💭	An interpretation of this book
📚	From the literature (not verified here)

The distinction between 🆕 and 📚 is the lifeline of this book. Writing so that a claim can be retracted is the only discipline of style observed here.

P-4. Two languages

The same content is written twice. Both are needed.

Language	Basic object	Good for
Configuration space	configurations of cell values $\Sigma^{\mathcal{L}}$	intuition, computation, pictures
Algebra	algebra of observables $\mathcal{A}$	theorems, constraints, region splitting, quantisation

Chapter 2 builds the bridge; afterwards the two are used as convenient. **The algebraic language is used not for compression but to get the attribution of obstacles right.**

Chapter 0 What a cellular automaton is

— How much variety can a simple rule generate

No prerequisites are needed for this chapter. A lattice, a rule, and time will do. Everything below is an actual run.

0-1. There are only three parts

To build a cellular automaton (CA) you fix three things.

(1) Lattice where the cells sit
(2) States the (finitely many) values a cell can take
(3) Rule a function of the neighbourhood giving the next value

And **every cell applies the same rule at the same time**. That is all.

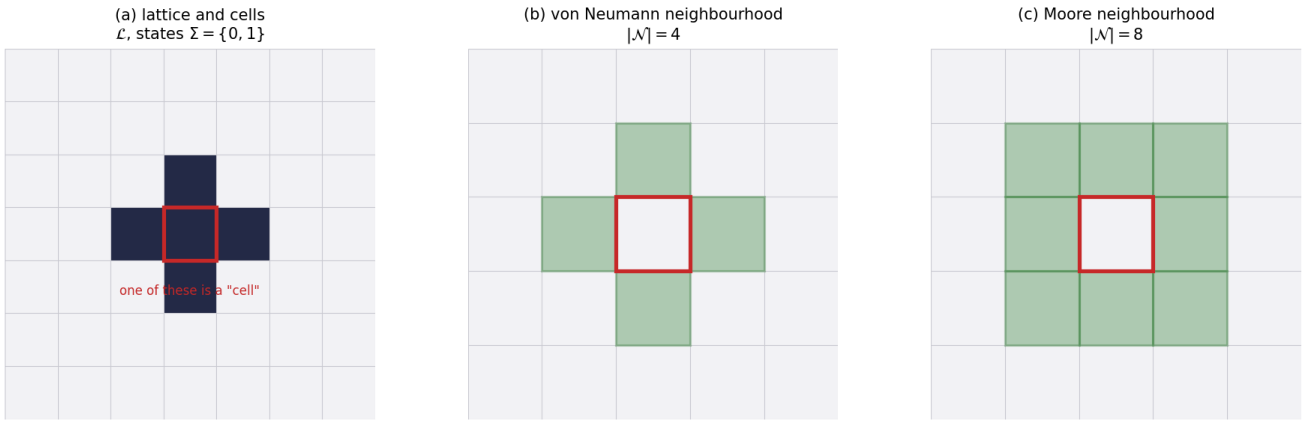


Figure 0-1. (a) Cells sit on a lattice, each carrying a value; here black $= 1$, white $= 0$. (b)(c) The neighbourhood can be chosen. Four cells up/down/left/right is the von Neumann neighbourhood; adding the diagonals gives the Moore neighbourhood.

The important point is that the rule **sees only the neighbourhood**. The value of a distant cell has no direct influence on what you become next. This single fact later becomes **causality** (§0-5).

The other important point is that the states are **finite in number** — the claim that one location holds only a finite amount of information. This later underwrites both "no ultraviolet divergence" and "$U(1)$ cannot be put in".

0-2. The Game of Life — from three lines

Conway's Game of Life uses a 2-dimensional lattice, two states, the Moore neighbourhood, and a rule of three lines.

A live cell survives if 2 or 3 neighbours are alive; otherwise it dies
A dead cell becomes alive if exactly 3 neighbours are alive
Otherwise nothing changes

Conventionally written **B3/S23** (Birth 3 / Survive 2,3). **That is the whole rule** — no hidden parameters, no randomness, no exceptions.

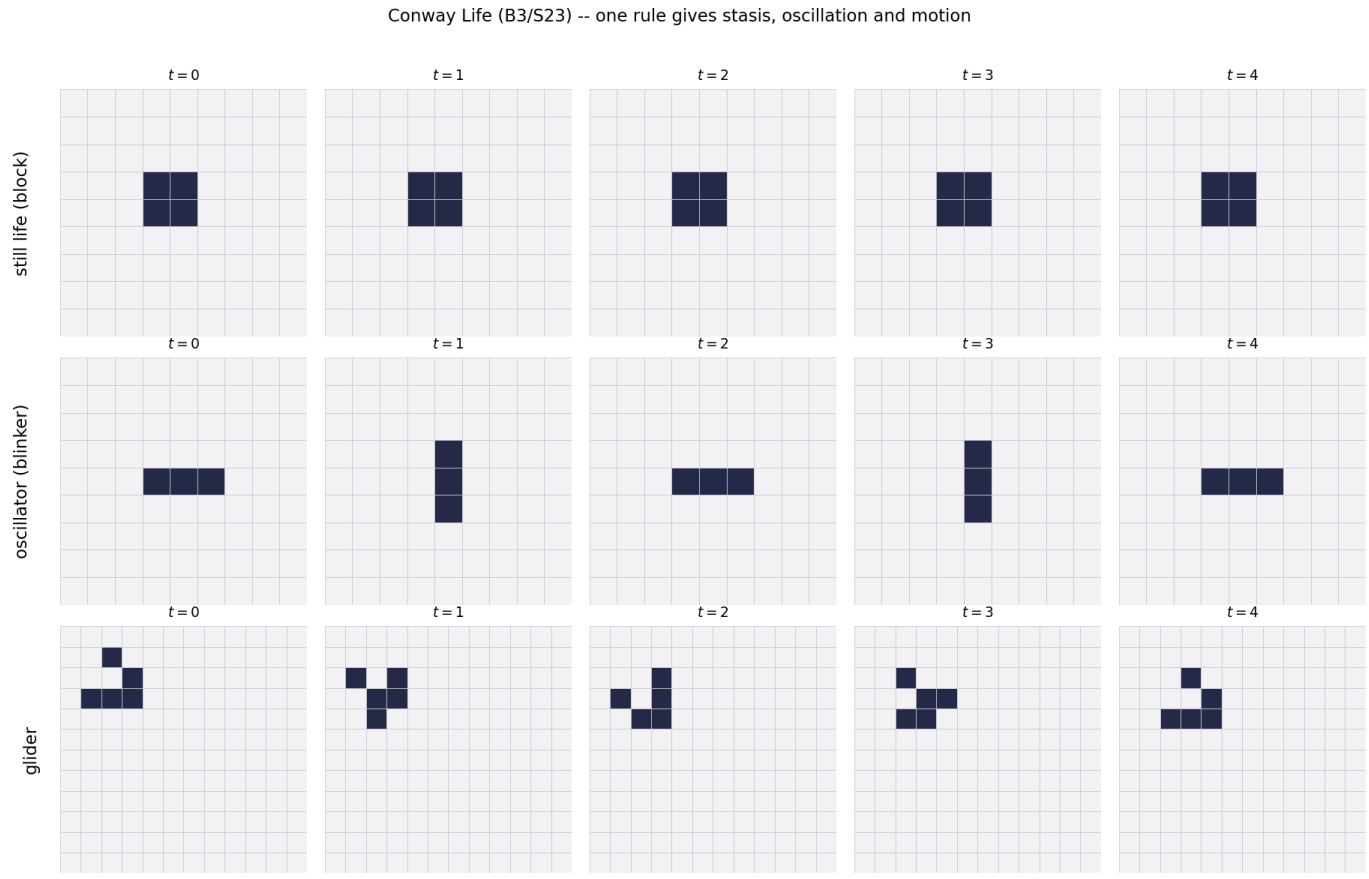


Figure 0-2. Three kinds of behaviour from one rule. Top: a still life (block) sits forever. Middle: an oscillator (blinker) returns with period 2. Bottom: a glider returns to its own shape after 4 steps, displaced one cell diagonally — a structure that moves.

That all three come from the same rule is the substance of the surprise. **Nobody designed "the stationary one", "the oscillating one" and "the moving one" separately. One rule already contained them.**

It is further known that gliders can be collided to build logic gates, and hence a **universal computer** 📚. Three lines of rule carry the full power of computation.

> 💭 This is where the book starts. **A simple rule need not have simple behaviour.** So perhaps the rule of our universe is simple too.

0-3. In one dimension you can exhaust every rule

Two dimensions makes beautiful pictures but has far too many rules to survey. Restrict to **one dimension, two states, a three-cell neighbourhood** and there are exactly

$$2^{2^3} = 2^8 = 256 \text{ rules.}$$

These are the **elementary cellular automata** (ECA), numbered 0 to 255.

Defining a rule means nothing more than writing one output bit for each of the 8 possible neighbourhoods. Read those 8 bits as a binary number and you have the rule number.

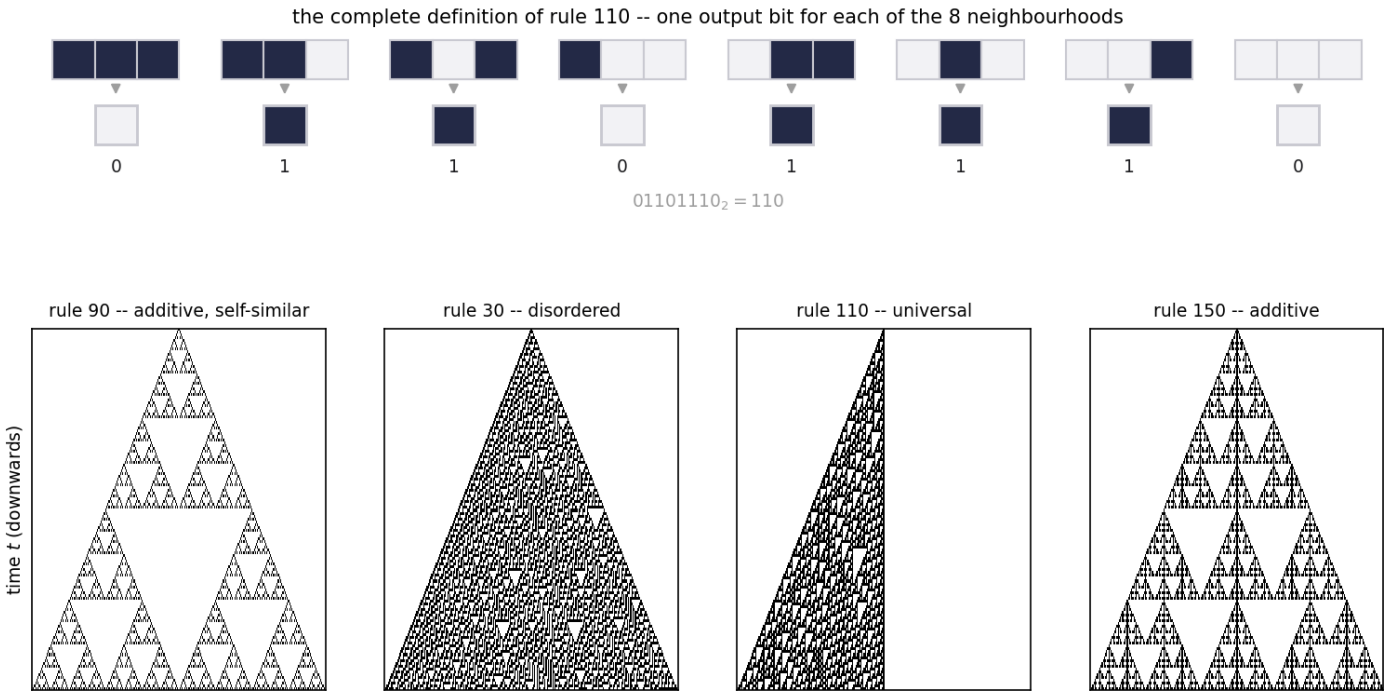


Figure 0-3. Top: the complete definition of rule 110. Fill in eight boxes and the rule is fixed; $01101110_2 = 110$ is its name. Bottom: space-time diagrams started from a single cell set to 1 ($t = 0$ at the top, time downwards). From one framework come fractals (90, 150), disorder (30), and a mixture of structure and background (110).

The bottom row of Figure 0-3 is this book in miniature. **The framework is one; what comes out differs in kind.**

Rule	Behaviour	Property
90, 150	Sierpinski-like fractal	**additive** (linear over $\mathbb{F}_2$); exactly solvable
30	disordered; used as a random source 📚	nonlinear; hard to predict
110	local structures move on a background and collide	**universal computation** 📚

📚 The universality of rule 110 is due to Cook (2004). **With a three-cell neighbourhood and two states — the minimal setting — one is already at the full power of computation.**

0-4. Four types 📚

Wolfram classified ECA behaviour into four classes.

Class	Behaviour	Example
1	settles to a uniform state	rules 0, 32
2	settles into simple periodic structure	rules 4, 108
3	stays disordered	rules 30, 45
4	local structures move about and interact	**rule 110**

⚠️ The classification has no rigorous definition and the boundaries are fuzzy. But the observation that **class 4 is tied to computational capability** matters later.

> 💭 Physically, class 4 is the interesting one. Classes 1 and 2 correspond to a universe where nothing happens; class 3 to a universe that cannot hold structure. **Structure that persists, and moves, and interacts** — that is exactly the condition our universe satisfies.

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0-5. How fast can information travel?

Here is the first step in reading a CA as physics. Because the rule **sees only the neighbourhood**, information advances at most one cell per step.

The check is simple. **Flip a single cell of the initial condition and follow the difference between the two worlds.**

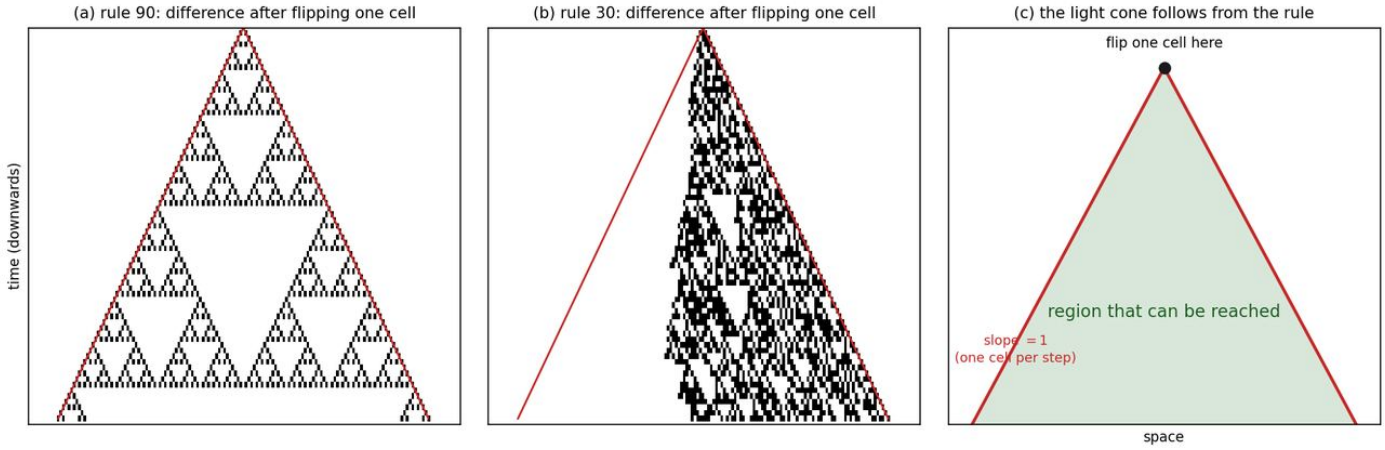


Figure 0-4. (a)(b) From a disordered initial condition, the difference between two worlds that differ by one flipped cell in the middle. The red lines mark the limit of one cell per step. The difference never crosses them. (c) That triangle is the light cone.

$$\boxed{\text{The light cone follows from the structure of the rule. It was not assumed.}}$$

In real physics causality is an **axiom imposed on the theory**; relativity requires that nothing outruns light. A CA needs no such requirement. **Because the neighbourhood is finite, the bound is there from the start.**

Note that the difference need not **fill** the cone. The cone is an upper bound; actual propagation can be slower. Only exceeding it is impossible.

0-6. Can time be run backwards?

Can the Game of Life be played in reverse? **No** — different pasts lead to the same present.

But a small rewriting makes any rule **exactly reversible**. The recipe is called the **second-order form**.

$$\boxed{x_{t+1} = f(x_t) \ \oplus \ x_{t-1}}$$

Remember not only the present but **the previous time step**, and XOR it with the rule's output. Since $\oplus$ undoes itself, the equation can be solved backwards:

$$x_{t-1} = f(x_t) \ \oplus \ x_{t+1}$$

The same rule is its own inverse.

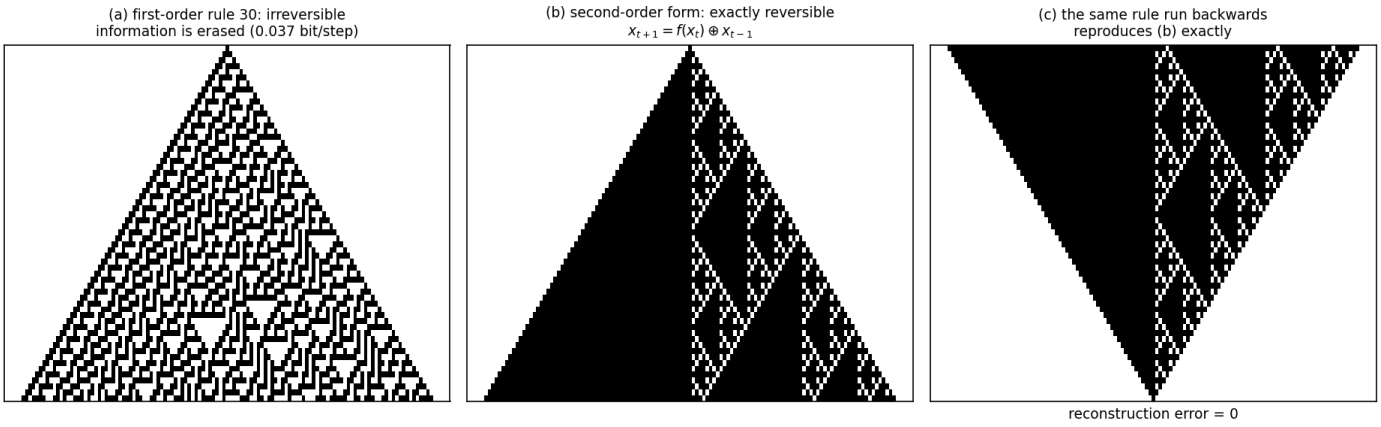


Figure 0-5. (a) First-order rule 30 is irreversible; information is erased. (b) The same f in second-order form is exactly reversible. (c) Starting from the last two rows of (b) and running backwards reproduces (b) exactly (reconstruction error $= 0$).

> Nothing about the rule was changed. **Only what we call the "state" was changed.**
> Reversibility is not a property of the rule but of the definition of the state.

This single point runs through the whole second half of the book. Chapter 15 measures, bit by bit, that a reversible CA has no logical erasure (for the relation to heat, §87-4); Chapter 17 shows that being reversible means having no ground state. Reversibility is not all good news.

0-7. Self-similarity — the rule contains itself

Additive rules (linear over $\mathbb{F}_2$, like 90 and 150) have a striking property.

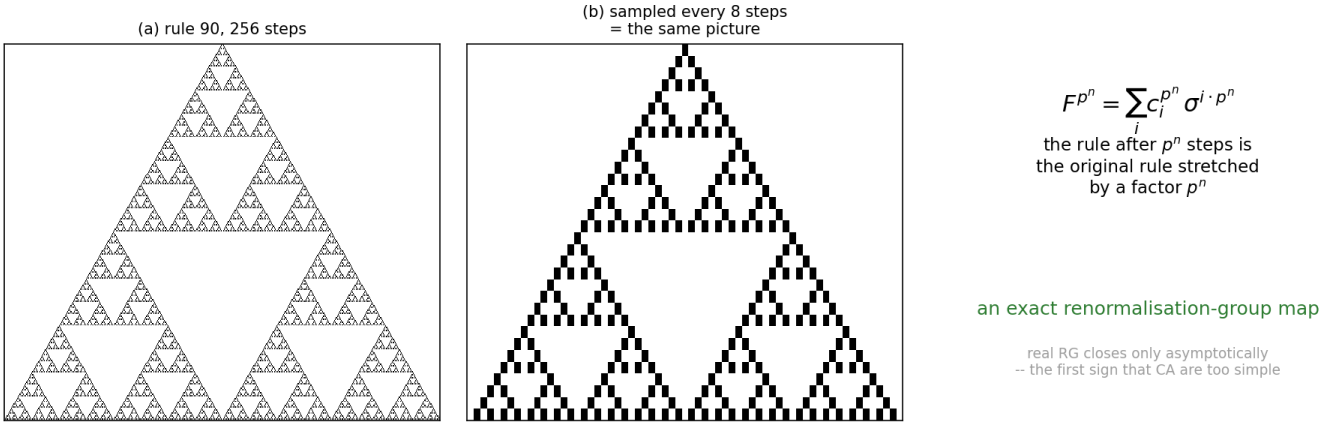


Figure 0-6. (a) Rule 90 run for 256 steps. (b) Sampled every 8 steps and every 8 cells. The same picture appears. (c) The reason.

$$F^{p^n} = \sum_i c_i^{p^n} \sigma^{i \cdot p^n}$$

The rule after p^n steps has the same form as the original rule stretched by a factor p^n. This is an exact **renormalisation-group transformation**.

The renormalisation group of real physics closes only approximately, near a fixed point. That it closes exactly here is **the first sign that CA are too simple**. In Chapter 10 that sign becomes a theorem: solvable models form an island of measure zero.

0-8. Why a physicist looks at this

The toolkit is now complete: **a lattice, finite states, a local rule, simultaneous update**. And already the following are in hand.

What is already there	From where
causality (light cone)	the neighbourhood is finite (§0-5)
finiteness	the states are finite (§0-1)
reversibility	redefine the state (§0-6)
exact renormalisation	for additive rules (§0-7)
complexity, universal computation	rule 110, the Game of Life (§0-2, §0-3)

Several basic properties of physics have appeared **without being assumed**. Perhaps the rest will too.

This book presses that expectation with forty-one numerical experiments and twelve further calculations. **The aim is to fix, to the digit, how far it goes and where it stops.**

Question 1	How far can a cellular automaton write physical law?
Question 2	What else is needed to describe the real world?
Question 3	Can the world be both simple and non-deterministic?

The next chapter restates all of this precisely. **Chapter 1 is Chapter 0 in mathematical language; no new concept is added.**

Part I Writing the rule — what exactly was assumed

Chapter 1 Making the definition precise

— Writing down exactly what was assumed, with no escape hatch

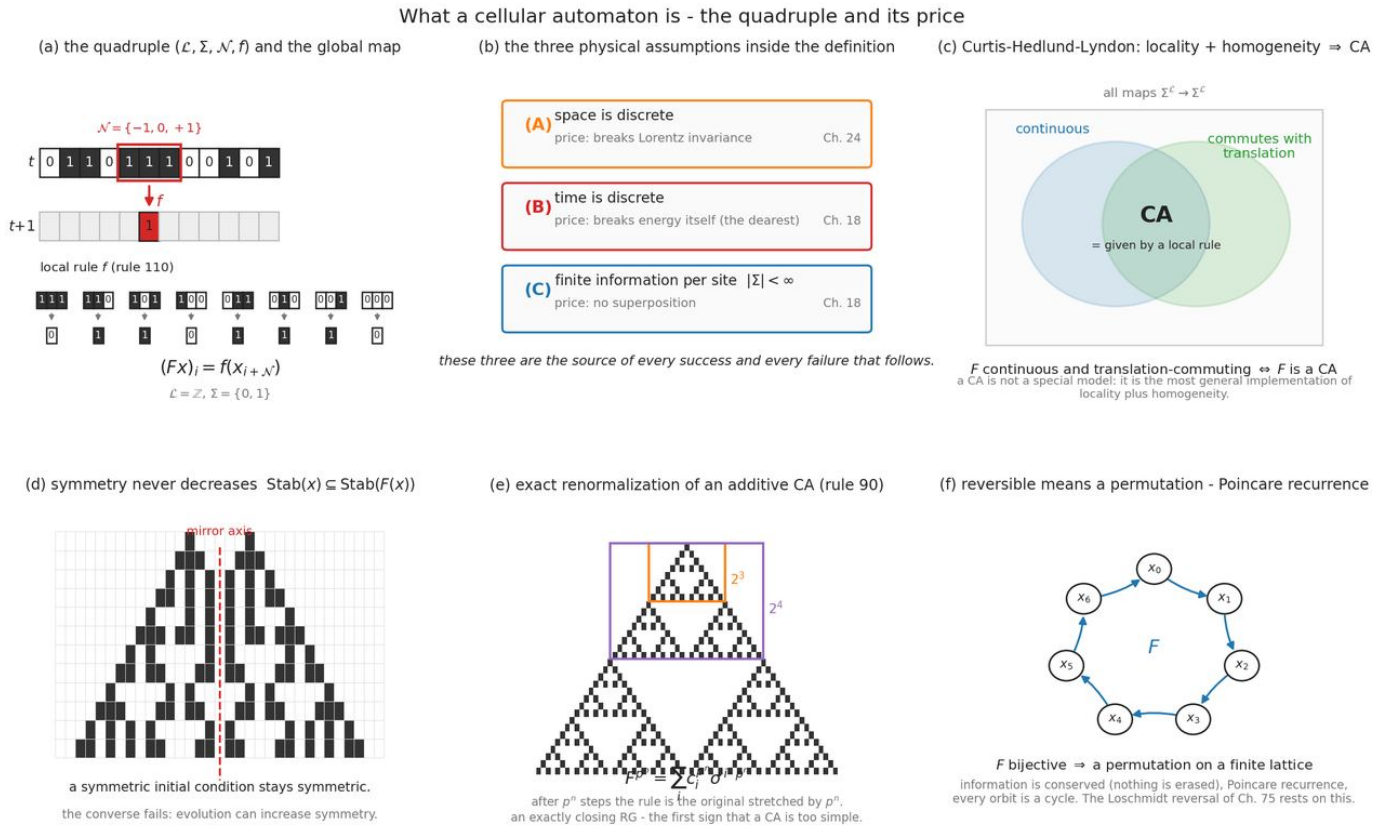


Fig. 1-1. (a) The quadruple $(\mathcal{L}, \Sigma, \mathcal{N}, f)$ and the global map $(Fx)_i = f(x_{i+\mathcal{N}})$; the local rule is a lookup table (rule 110). (b) The three physical assumptions already inside the definition, and what each costs. The dearest is (B), discrete time. (c) The Curtis-Hedlund-Lyndon theorem: impose locality and homogeneity and there is nothing but a CA. (d) Symmetry never decreases - a symmetric initial condition stays symmetric. (e) Exact renormalization of an additive CA (rule 90): after p^n steps the rule is the original stretched by p^n. (f) Reversible means a permutation on a finite lattice, from which Poincare recurrence follows.

Chapter 0 restated in mathematics. **No new concept is added.** Only precision.

1-1. Definition

$$(\mathcal{L},\ \Sigma,\ \mathcal{N},\ f)$$

Symbol	Meaning
$\mathcal{L}$	lattice (typically $\mathbb{Z}^d$)
Σ	the **finite** set of states of one cell
$\mathcal{N}$	neighbourhood (finitely many offsets)
$f : \Sigma^{\mathcal{N}} \to \Sigma$	local rule

Global evolution: $(Fx)_i = f(x_{i+\mathcal{N}})$.

Three physical claims are already built in. **These three are the source of every success and every failure that follows.**

$$\begin{aligned}&\text{(A) space is discrete}\\&\text{(B) time is discrete}\\&\text{(C) one location holds finite information}\quad(|\Sigma|<\infty)\end{aligned}$$

Stated in advance: **(B) is the most expensive** (Chapter 17). (A) breaks Lorentz invariance; (B) breaks **energy itself**.

1-2. The Curtis–Hedlund–Lyndon theorem ✅

$$\boxed{F \text{ continuous and shift-equivariant} \iff F \text{ comes from a local rule}}$$

The physical reading is the starting point of the book. **Impose "local" and "homogeneous" and there is no alternative to a CA.** A CA is not a special model; it is the most general implementation of locality plus homogeneity.

📚 Hedlund (1969).

> ⚠️ **Restriction (thirteenth edition).** The Curtis–Hedlund–Lyndon theorem assumes more than locality and homogeneity: the full shift, the product topology, continuity and a finite alphabet. "Nothing but a CA" holds within this frame. §87-9.

1-3. Symmetry never decreases ✅

$$\mathrm{Stab}(x) \subseteq \mathrm{Stab}(F(x))$$

A symmetry that leaves a configuration invariant leaves its successor invariant. That the Game of Life keeps symmetric patterns from symmetric initial conditions is a consequence of this one line. The converse fails — evolution can **increase** symmetry. That matters later for the arrow of time.

1-4. Exact renormalisation for additive CA ✅

For rules linear over $\Sigma = \mathbb{F}_p$, $F = \sum_i c_i \sigma^i$, and by Frobenius

$$F^{p^n} = \sum_i c_i^{p^n} \sigma^{i \cdot p^n}$$

Real renormalisation groups close only approximately. Exact closure here is **the first sign that CA are too simple**.

1-5. Reversibility

If F is a bijection the CA is reversible. On a finite lattice it is a **permutation**, and everything below follows.

Consequence	Content
information preserved	nothing erased (quantified in Chapter 15)
Poincaré recurrence	every orbit is a finite cycle
no ground state	the spectrum of a permutation is a set of quasi-energies on the unit circle (its relation to energy is restricted in §87-7)

The **second-order form** makes any rule reversible without touching its content.

$$x_{t+1} = f(x_t) \oplus x_{t-1}$$

Reversibility is not a property of the rule but of what one calls the state.

> ⚠️ **Restriction (thirteenth edition).** Using $\oplus$ requires choosing a group on the state set (any finite set can carry $\mathbb{Z}_{|\Sigma|}$). And the second-order form is not an embedding of the original rule but **a different reversible dynamics** (for rule 30 it agrees with the first-order orbit on the first step only; K432). §87-4.

Chapter 2 The triple — algebra, automorphism, state

— What are the minimum parts needed to build a physical theory

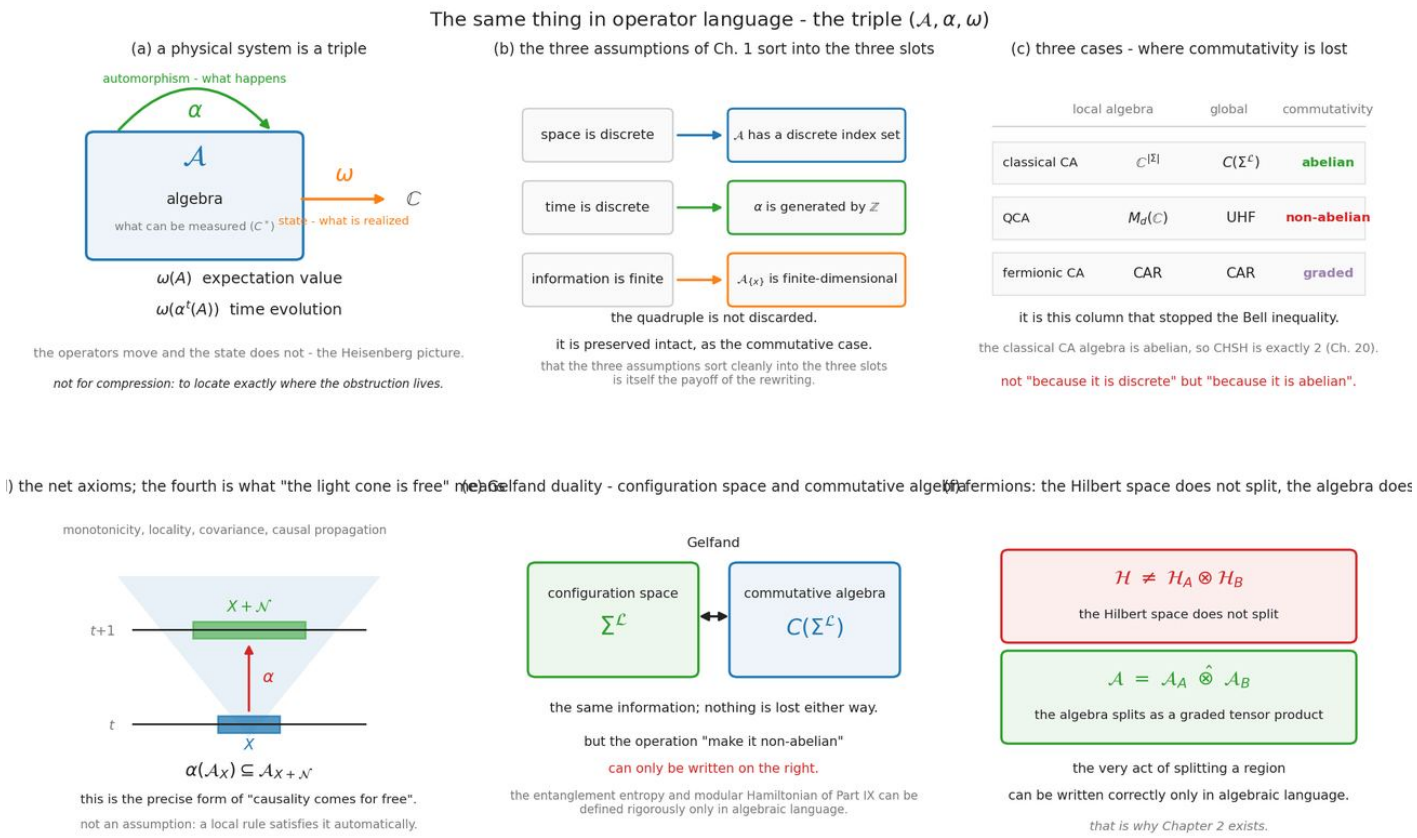


Fig. 2-1. (a) A physical system is the triple $(\mathcal{A}, \alpha, \omega)$; the operators move and the state does not. (b) The three assumptions of Chapter 1 sort cleanly into the three slots; the quadruple survives as the commutative case. (c) The three cases. Where commutativity is lost is where the Bell inequality was stopped. (d) The net axioms; the fourth, causal propagation $\alpha(\mathcal{A}_X) \subseteq \mathcal{A}_{X+\mathcal{N}}$, is the precise form of "the light cone comes for free". (e) Gelfand duality: configuration space and commutative algebra carry the same information, but the operation "make it non-abelian" can be written only on the algebra side. (f) For fermions the Hilbert space does not split, while the algebra splits as a graded tensor product.

Chapter 1's four-tuple is the language of configuration space. The same thing is now written in operators. **Not for compression, but to locate exactly where the obstacles live.**

2-1. The triple

$$\boxed{\text{physical system} = (\mathcal{A},\ \alpha,\ \omega)}$$

Symbol	Name	Content
$\mathcal{A}$	algebra	C^*-algebra of observables (what can be measured)
α	automorphism	time evolution $\mathcal{A} \to \mathcal{A}$ (what happens)
ω	state	normalised positive functional (what is realised)

Expectations are $\omega(A)$; evolution is $\omega(\alpha^t(A))$. **Operators move, states do not** — the Heisenberg picture.

The three assumptions of Chapter 1 fall neatly into three slots.

$$\begin{aligned} \text{space discrete} &\longrightarrow \text{the index set of } \mathcal{A} \text{ is discrete} \\ \text{time discrete} &\longrightarrow \alpha \text{ is generated by a } \mathbb{Z}\text{-action} \\ \text{finite information} &\longrightarrow \mathcal{A}_{\{x\}} \text{ is finite-dimensional} \end{aligned}$$

2-2. Three cases

$$\mathcal{A} = \overline{\bigcup_{X \text{ finite}} \mathcal{A}_X}$$

	Local algebra	Whole	Commutativity
classical CA	$\mathbb{C}^{\lvert\Sigma\rvert}$	$C(\Sigma^{\mathcal{L}})$	**commutative**
QCA	$M_d(\mathbb{C})$	UHF algebra	non-commutative
fermionic CA	CAR	CAR algebra	$\mathbb{Z}_2$-graded

The classical case is exactly Gelfand duality: **configuration space and commutative algebra carry the same information**. The four-tuple is not discarded but preserved as the commutative case.

Splitting a fermionic system into regions (essential in Part IX) works differently:

$$\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B, \qquad \mathcal{A} = \mathcal{A}_A \hat{\otimes} \mathcal{A}_B$$

The Hilbert space does not factorise but **the algebra does, as a graded tensor product**. The entanglement entropies and modular Hamiltonians of Part IX are rigorously definable only in the algebraic language.

> ⚠️ **Restriction (thirteenth edition).** The Fock space of finitely many modes can be factorized as a tensor product once the modes are ordered (Jordan–Wigner). The issue is that the local algebras are graded and this factorization is not local. §87-9.

2-3. The net axioms (discrete Haag–Kastler)

Axiom	Content	Meaning for a CA
isotony	$X \subseteq Y \Rightarrow \mathcal{A}_X \subseteq \mathcal{A}_Y$	larger regions contain more
locality	$X \cap Y = \varnothing \Rightarrow [\mathcal{A}_X, \mathcal{A}_Y] = 0$	separated regions are jointly measurable
covariance	$\sigma_v(\mathcal{A}_X) = \mathcal{A}_{X+v}$	translation symmetry
causal propagation	$\alpha(\mathcal{A}_X) \subseteq \mathcal{A}_{X+\mathcal{N}}$	**light cone**

The fourth axiom is the precise form of "causality comes for free". It is not an assumption but a property any local rule automatically has.

2-4. CHL and GNVW side by side ✅📚

The definition of locality becomes literally identical for classical and quantum.

$$\boxed{\alpha \text{ is local} \iff \alpha(\mathcal{A}_X) \subseteq \mathcal{A}_{X+\mathcal{N}}}$$

	classical CA	QCA
algebra	commutative	non-commutative
evolution	$*$-endomorphism	$*$-automorphism
locality	as above	**identical**
characterisation	**CHL** (1969)	**GNVW** (2012) 📚
1D structure theorem	block permutations ∘ shift 📚	circuit ∘ shift (index) 📚
invariant	—	index $\mathrm{ind} \in \mathbb{Q}_{>0}$

CHL and GNVW are **the same theorem differing in one point: commutativity**.
Even in quantum theory, locality plus homogeneity does not let you leave CA structure.

> 💭 Chapter 18 reaches the fork "classical CA or QCA". **Both sides of that fork are CA.** In the algebraic language, "maybe a CA cannot write the universe" is hard even to formulate.

⚠️ In higher dimensions, reversibility of 2D classical CA is undecidable, and non-trivial 3D QCA exist (📚 Kari 1994; Freedman–Hastings 2020; Haah–Fidkowski–Hastings). Not verified here.

Part II What comes for free

Count the granted part of the wish first. Everything listed here follows from a local rule with no assumption beyond the triple of Chapter 2 (including the CAR algebra and the complex structure) — a restriction of scope made in the eleventh edition, §85-4.

Chapter 3 Causality and the light cone ✅

— Why is there a speed limit on information

In real physics the causal structure follows from the metric, and causality is imposed as an axiom.

In a CA, **the rule sees only the neighbourhood, so information advances one cell per step**.

> The light cone is not an axiom; it follows from the structure of the rule.

Moreover, making the rule space-dependent lets one **read off a metric**.

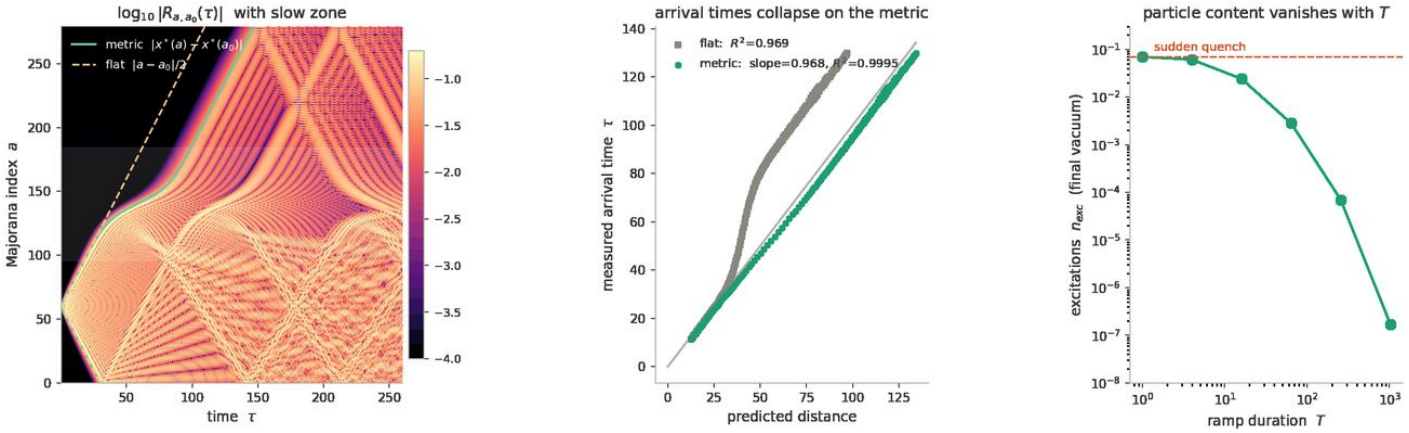


Figure 3-1. Reconstructing the metric of a curved spacetime from local propagation speeds. Slope 1.00 ± 0.04, $R^2 = 0.999$.

Making the rule time-dependent produces expansion ($\beta \neq 0$). **Kinematic general relativity is completely reproduced by a CA.**

> ⚠️ **Restriction (thirteenth edition).** What local propagation speeds define is an effective causal structure. The conformal factor, curvature, clock calibration and dynamics are further input. "Completely reproduced" and "reconstructs the metric" are withdrawn. §87-9.

Chapter 4 Creation and annihilation operators are a complex structure ✅

— Why are particle, vacuum and uncertainty the same single object

The most important identification in the book, and the bridge between CA and quantum theory.

4-1. The identification

In Majorana form, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. Choosing "which pair of γ's to call one particle" is nothing other than choosing a **complex structure** J on a real $2N$-dimensional space.

$$a_j = \frac{\gamma_{2j-1} + i\gamma_{2j}}{2}$$

The set of all such choices is a homogeneous space.

$$\boxed{\mathcal{M} = O(2N)/U(N), \qquad \dim \mathcal{M} = N(N-1)}$$

The covariance matrix of a pure Gaussian state ($\langle \gamma_a \gamma_b \rangle = \delta_{ab} + i\Gamma_{ab}$) satisfies $\Gamma^2 = -\mathbb{1}$: Γ is itself a complex structure.

4-2. Three faces 💭

$$\boxed{\mathcal{M} = \{\text{saturated uncertainty relations}\} = \{\text{vacua}\} = \{\text{fillings of the Dirac sea}\}}$$

Reading	Content
uncertainty	$\|\Gamma\| \le 1$ is the fermionic uncertainty relation; pure Gaussian is equality
vacuum	each Γ fixes what counts as a particle, hence a vacuum
Dirac sea	the choice of how to fill the negative-energy states

"What is a particle", "what is the vacuum" and "what is the uncertainty relation" are three names for one geometric object.

In algebraic language $\mathcal{M}$ lives in the third slot: **the pure quasi-free states of the CAR algebra**.

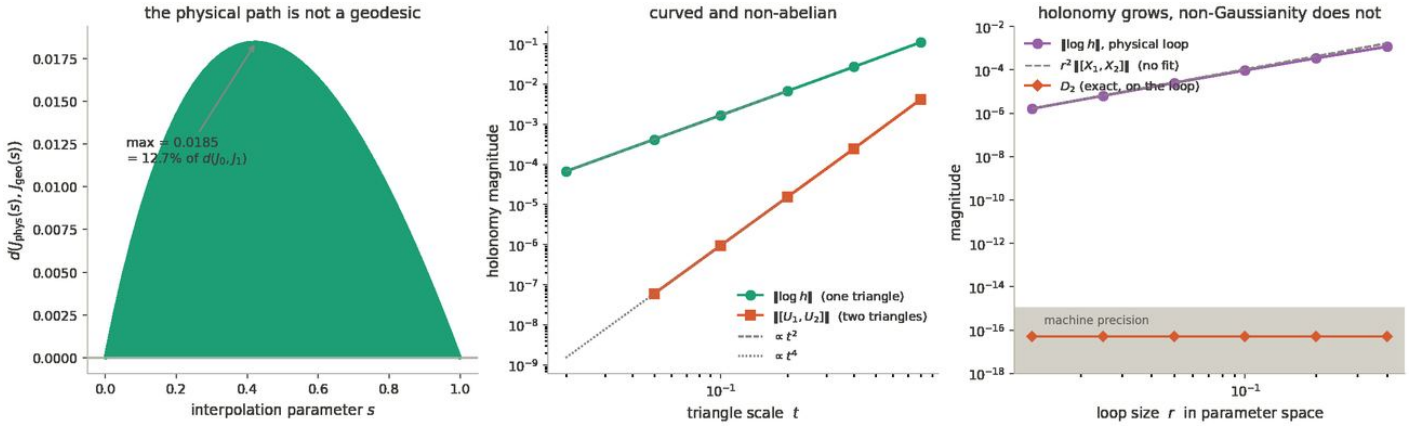


Figure 4-1. Transporting J over $\mathcal{M}$. A closed loop produces a $U(N)$ holonomy. Holonomy exponent 2.0002, alignment 1.000000.

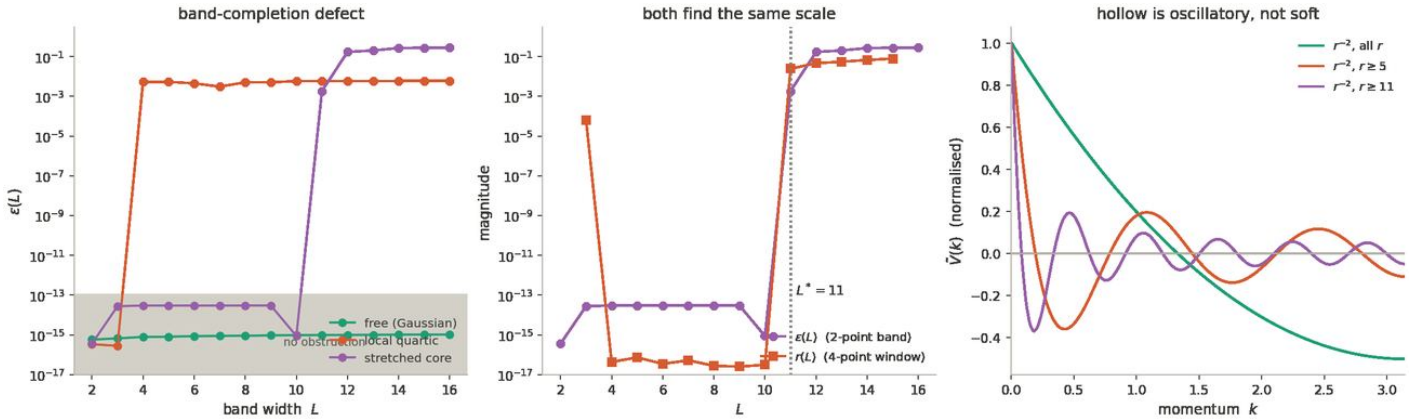


Figure 4-2. Gluing and saturation of uncertainty. Equality in $\|\Gamma\| \le 1$ is the pure Gaussian case.

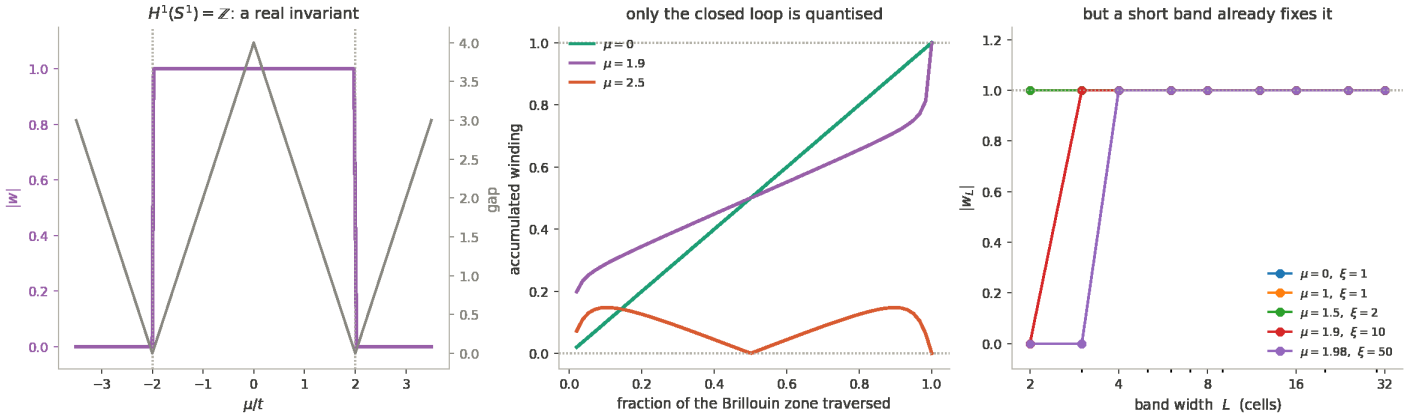


Figure 4-3. Transporting J around a ring; topology restricts the admissible configurations.

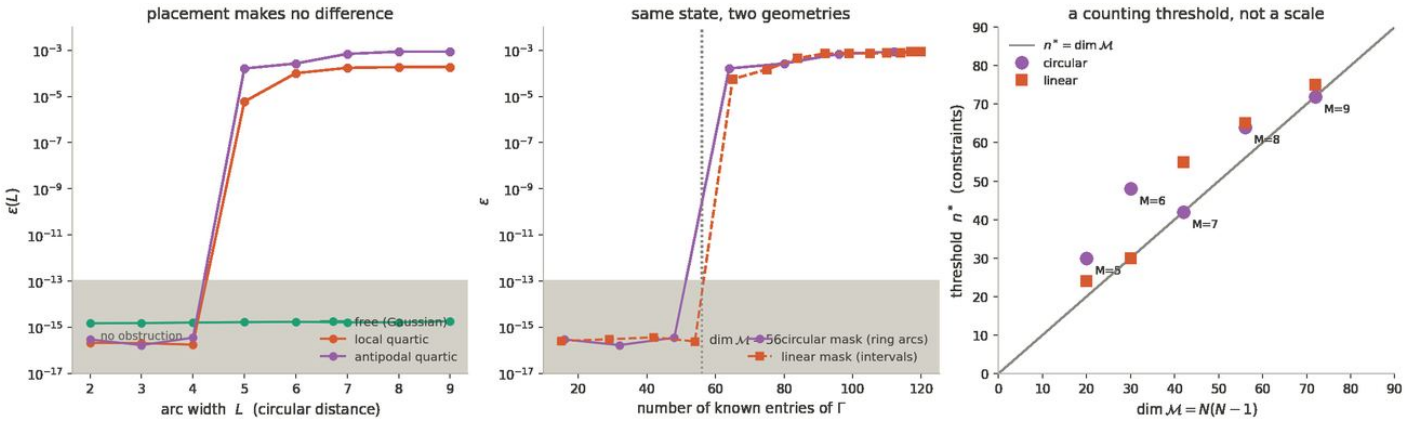


Figure 4-4. Level structure on a ring: boundary conditions fix the discrete spectrum.

> ⚠️ **Restriction (thirteenth edition).** The correspondence is exact for the geometry of Gaussian states (Γ and the complex structure, the Fock vacuum). "Dirac sea" and "negative energy" need a choice of Hamiltonian. §87-9.

Chapter 5 Zero-point motion ✅

— Why is the vacuum not at rest

$$E_0 = \tfrac{1}{2}\hbar\omega, \qquad \Delta x\,\Delta p \geq \frac{\hbar}{2}$$

Chapter 4's $\mathcal{M}$ is already the answer. $\Gamma^2 = -\mathbb{1}$ is the **saturation condition** of the uncertainty relation, and even at saturation γ_{2j-1} and γ_{2j} cannot be simultaneously sharp.

The ground-state energy $E_0 = -\frac{1}{2}\sum_k \epsilon_k$ carries the $\frac{1}{2}$ of zero-point motion. **It comes out with no assumption beyond the $\mathcal{M}$ of Chapter 4** (§85-4).

> ⚠️ **Restriction (thirteenth edition).** The opening $E_0 = \frac{1}{2}\hbar\omega$ is the bosonic formula. The fermionic ground-state energy is negative, $-\frac{1}{2}\sum_k \epsilon_k$ (with the ordering $H = \sum_k \epsilon_k(a_k^\dagger a_k - \frac{1}{2})$). The $\rho_{\text{vac}} \sim \sum \epsilon_k/2$ of the next section is an estimate of size and does not include the sign. §87-9.

5-1. But the cosmological constant problem is inherited ❌

$$\rho_{\text{vac}} \sim \sum_k \frac{\epsilon_k}{2} \sim \Lambda_{\text{cutoff}}^4$$

The lattice removes the divergence but **only fixes the cutoff at the lattice spacing; it does not make the result small**. At the Planck length one is 10^{120} times the observed value.

> A CA avoids the infinity but not the cosmological constant problem. The problem is the **size**, not the divergence.

Chapter 6 Mass is geometry ✅

— What kind of quantity is mass — and where does the de Broglie wave come from

In real physics $E = mc^2$ is a derived theorem.

6-1. Mass is a ratio of lengths

In a CA, varying m and the velocity parameter v independently leaves the zero mode unchanged **as long as m/v is fixed**.

Test	Result
m varied by 56%, v by 80%, m/v fixed	wavefunction correlation 0.9997
localisation length vs exact lattice formula	agrees to 0.01%

$$\xi = \frac{1}{\ln(1 + m/v)} \longrightarrow \frac{v}{m} = \frac{\hbar}{mc}$$

$$\boxed{\text{CA mass} = \frac{\text{lattice spacing}}{\text{Compton wavelength}}}$$

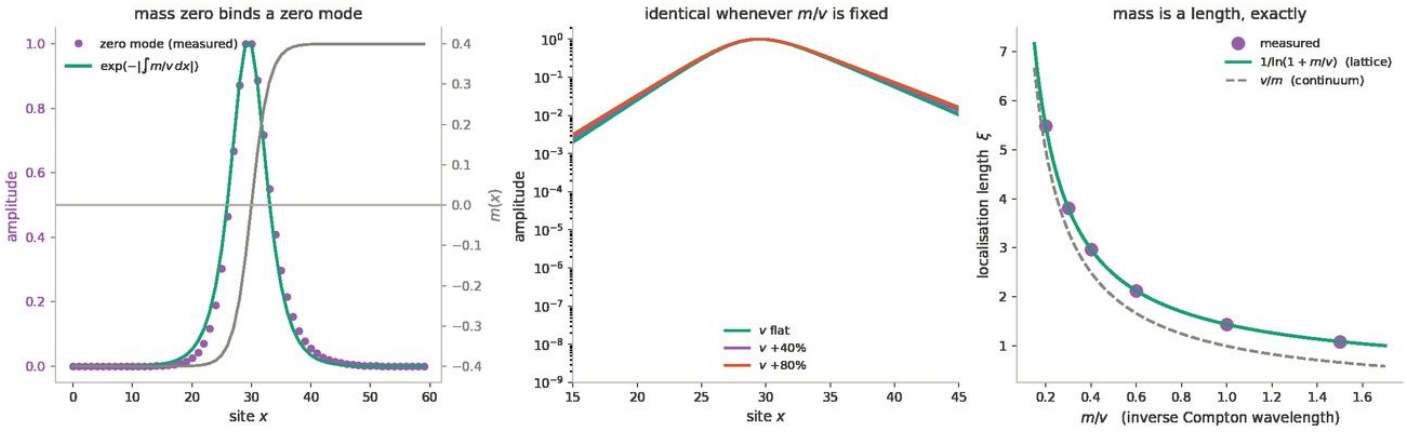


Figure 6-1. Independent changes of m and v leave the zero mode fixed when m/v is fixed.

> 💭 **A CA does not prove $E = mc^2$; it has no language in which to restate it.** There is no separate entity to convert: mass is a ratio of lengths from the start, so the conversion formula degenerates into an identity.
>
> But **the ratio of masses remains an irreducible input**. Part VIII digs into that.

6-2. The dispersion relation is a spherical triangle 🆕

If mass is a ratio of lengths, then **what does that same ratio look like when the particle is not at rest?** The answer is the de Broglie wave, and it needs no new assumption.

Take the Dirac quantum walk (Bialynicki-Birula; the same object as Feynman's checkerboard). One step consists of two operations: at each site rotate the two components by the mass angle θ, $C(\theta) = e^{-i\theta\sigma^1}$; then shift the upper component one site right and the lower one site left. In momentum space $U(k) = \mathrm{diag}(e^{-ik}, e^{+ik})C(\theta)$, with $\det U = 1$ and $\mathrm{tr}\, U = 2\cos k \cos\theta$, so

$$\cos\omega = \cos k \cdot \cos\theta$$

This is the Pythagorean theorem for a right triangle on a sphere, with legs k, θ and hypotenuse ω. In the continuum limit $\omega^2 \simeq k^2 + \theta^2$, i.e. $E^2 = p^2c^2 + m^2c^4$ — the **flat** triangle (the deviation vanishes quadratically; measured exponent 2.040).

Lattice physics is spherical trigonometry; continuum physics is plane trigonometry.

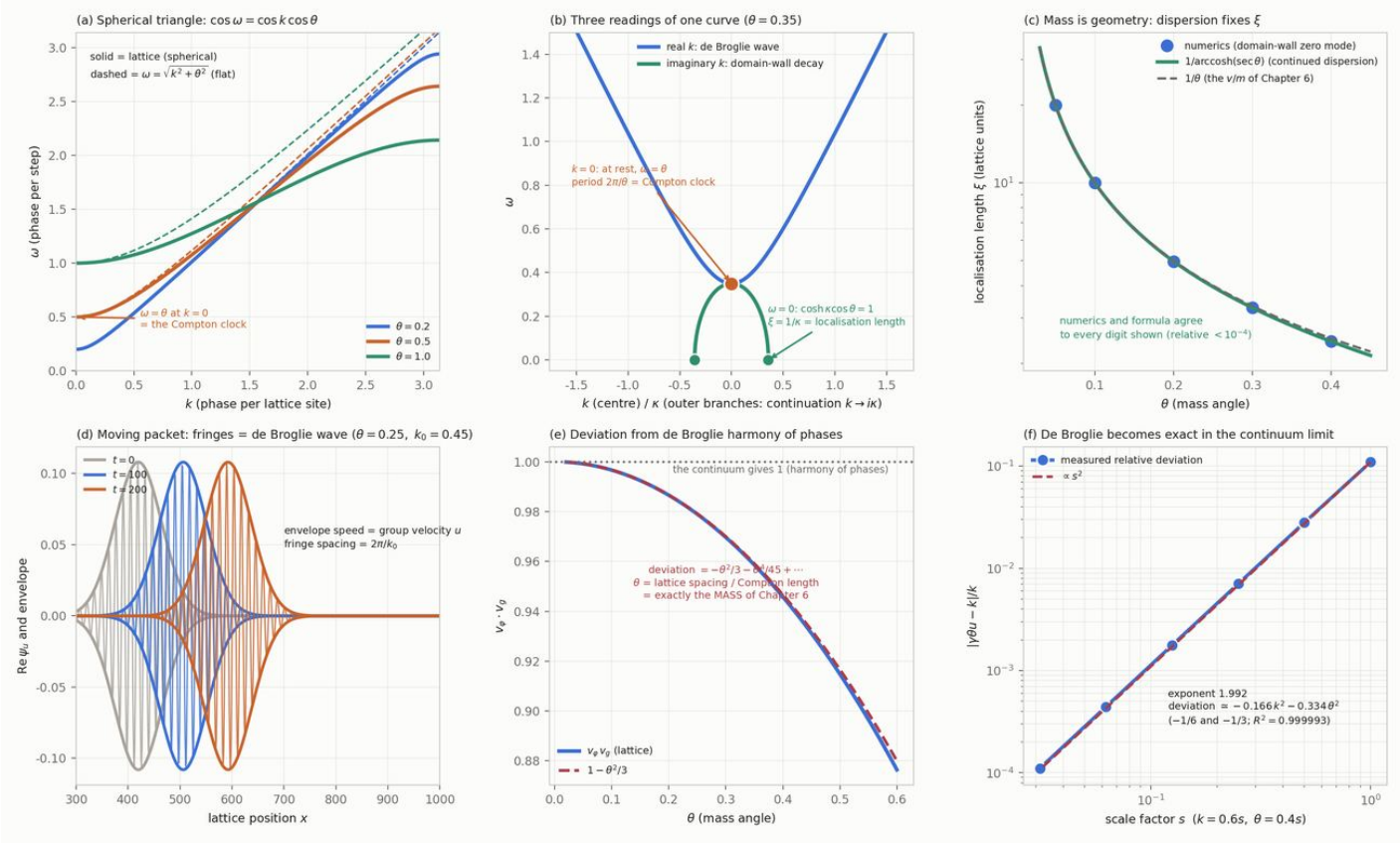


Figure 6-2. (a) The dispersion is a spherical triangle. (b) Three ways to read one curve. (c) The localisation length is fixed by the dispersion. (d) Fringes of a moving packet are the de Broglie wave. (e)(f) The lattice deviation is the square of the mass.

6-3. Three ways to read one curve 🆕

A **single** curve, $\cos\omega = \cos k \cos\theta$, yields three different physical quantities depending on where in k it is read.

(i) $k = 0$ — the Compton clock. A particle at rest does not move, but its phase does: $\omega = \theta$, so the period is $T_C = 2\pi/\theta$ steps. This is what de Broglie called the particle's internal clock in his 1924 thesis; in the continuum, $\nu_0 = mc^2/h$.

(ii) $\omega = 0$, $k \to i\kappa$ — the domain-wall localisation length. The zero mode measured in 6-1 is nothing but the same dispersion continued to **imaginary momentum**. From $1 = \cosh\kappa\cos\theta$,

$$\xi = \frac{1}{\operatorname{arccosh}(\sec\theta)}$$

Building a domain wall numerically and measuring the decay reproduces this **to every digit displayed**.

θ	0.05	0.10	0.20	0.30	0.40
measured ξ	19.9917	9.9833	4.9666	3.2830	2.4324
$1/\operatorname{arccosh}(\sec\theta)$	19.9917	9.9833	4.9666	3.2830	2.4324
$1/\theta$ (continuum)	20.0000	10.0000	5.0000	3.3333	2.5000

As $\theta \to 0$, $\xi \to 1/\theta$ — the $v/m = \hbar/mc$ of 6-1, the **Compton wavelength**.

(iii) real k — the de Broglie wave. Read at real k, the same curve gives a spatially oscillating wave of wavelength $\lambda_{dB} = 2\pi/k$.

The Compton length is the reading at imaginary momentum; the de Broglie wavelength is the reading at real momentum. The mass θ is the angle joining them.

6-4. The derivation — looking at a clock at rest from an angle 🆕

Why are (i) and (iii) connected? De Broglie's own argument becomes concrete enough to draw on a CA.

A particle at rest turns its phase by θ **simultaneously** at every site; the surfaces of constant phase are horizontal lines perpendicular to the time axis. Boosting with rapidity η gives $(\omega, k) = (\theta\cosh\eta,\ \theta\sinh\eta)$ and velocity $u = \tanh\eta$, so

$$k = \theta\gamma u \qquad \Longrightarrow \qquad \boxed{p = \gamma m u, \quad \lambda_{\mathrm{dB}} = \frac{2\pi}{k} = \frac{h}{p}}$$

The surfaces of constant phase have merely tilted. What was "the same phase everywhere at once" in the rest frame acquires a spatial phase gradient in the moving frame, because simultaneity has changed. Its spatial period is the de Broglie wavelength. The phase velocity $v_\varphi = \omega/k = 1/u$ (restoring units, c^2/u) exceeds the particle's speed, but the product with the group velocity is $v_\varphi v_g = 1$ — de Broglie's **harmony of phases**. Since information travels only at the group velocity, there is no contradiction.

We ran the packet on the CA (Fig. 6-2 (d)). The envelope velocity agrees with $d\omega/dk$ of the chosen branch to 0.34%, and the fringe spacing equals $2\pi/k_0$. **Envelope (particle) and fringes (wave) are visibly two aspects of one solution.**

> ⚠️ **Restriction (thirteenth edition).** This boost does not come from a lattice symmetry. This section is motivation, not derivation: an approximate agreement at small k, θ. The exact formula is in §87-5.

6-5. How badly does the lattice break the dispersion relation — and the answer is the mass 🆕 (heading corrected in the eleventh edition; §6-8)

Here the CA makes a prediction the continuum does not. **On a lattice the de Broglie relation is not exact.**

$$v_\varphi v_g = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45} + \cdots, \qquad \frac{\gamma\theta u - k}{k} \simeq -\frac{k^2}{6} - \frac{\theta^2}{3} \quad (R^2 = 0.999993)$$

Shrinking k and θ together, the deviation vanishes as s^2 (measured exponent 1.992). Its size is θ^2, and

$$\theta = \frac{\text{lattice spacing}}{\text{Compton wavelength}} = \frac{a\,mc}{\hbar}$$

is **exactly the ratio that 6-1 called the mass**. Hence

$$\boxed{\text{deviation from the de Broglie relation } \sim \ (\text{mass})^2}$$

The heavier the particle, the more the de Broglie relation breaks. With the lattice spacing at the Planck length, an electron gives $\theta \sim 4 \times 10^{-23}$ and a deviation $\sim 10^{-45}$ — hopeless to observe, but it joins the short list in Part XI of **places where the lattice spacing shows its face**.

> ⚠️ **Correction (thirteenth edition).** Exactly, $(\gamma\theta u - k)/k = \theta\cot\theta\,(\sin k/k) - 1$ (K430). $-k^2/6$ has nothing to do with mass and survives at $\theta \to 0$, so the box "deviation $\sim$ (mass)²" is right only for $k \ll \theta$. The expansion of $v_\varphi v_g$ is that of $\theta\cot\theta$, taking $k \to 0$ first. §87-5.

6-6. Summary

One curve, $\cos\omega = \cos k\cos\theta$, gives the Compton clock at $k = 0$, the localisation length at $k \to i\kappa$, and the de Broglie wave at real k. The mass θ is the angle joining all three.

What was said of $E = mc^2$ holds for de Broglie too. **A CA does not derive the de Broglie relation. It has it.** Once mass is a ratio of lengths, reading that ratio at rest gives the Compton length and reading it in motion gives the de Broglie wavelength — the conversion formula degenerates into an identity again. Only one thing differs: in the continuum $\lambda = h/p$ is exact, while **on a CA it carries a correction of order** $(\text{mass})^2$.

> **The mass also fixes how exact the de Broglie relation is.**

12 calibrations declared, 4 failed. All four were faults in how I measured (a branch sign convention, thresholds set at fixed θ, and a "continuum limit" that shrank only k). Chasing those failures produced the coefficients $-\theta^2/3$ and $-k^2/6$, and hence the conclusion of 6-5.

6-7. Each component satisfies a discrete Klein–Gordon equation 🆕

— The first-order two-component walk and the second-order one-component rule are the same thing

The $U(k) = \mathrm{diag}(e^{-ik}, e^{+ik})\, e^{-i\theta\sigma^1}$ of §6-2 is a 2×2 matrix with $\det U = 1$ and $\mathrm{tr}\, U = 2\cos k \cos\theta$. By Cayley–Hamilton, $U^2 - (\mathrm{tr}\, U)\, U + (\det U)\, \mathbb{1} = 0$, that is

$$\boxed{U + U^{-1} = 2\cos k \cos\theta\, \mathbb{1}}$$

holds **identically**. The point is that the right-hand side is proportional to the identity: the same second-order relation is imposed on both components. Returning to position and time, $e^{\pm ik}$ is a shift to a neighbour and $U^{\pm 1}$ is one step forward or back in time, so **each component** $\phi \in \{u, d\}$ independently satisfies

$$\boxed{\phi_j^{\,n+1} = \cos\theta\, (\phi_{j+1}^{\,n} + \phi_{j-1}^{\,n}) - \phi_j^{\,n-1}}$$

The next value is fixed by the neighbours' present values and one's own previous value alone.

6-7-1. Calibration 🆕

A Dirac walk with random complex initial conditions was run for 30 steps at $L = 128$, and the difference from the recurrence taken for both components at every step.

θ	max. deviation from recurrence	max. group velocity (numerical)	$\cos\theta$	$\omega(k=0)$
0.05	1.3×10^{-15}	0.998750	0.998750	0.050000
0.1	1.1×10^{-15}	0.995004	0.995004	0.100000
0.3	1.1×10^{-15}	0.955336	0.955336	0.300000
1.0	9.9×10^{-16}	0.540302	0.540302	1.000000
1.5	9.0×10^{-16}	0.070737	0.070737	1.500000

The recurrence holds to machine precision (K403). The third and fourth columns are used in 6-7-3.

6-7-2. Three readings

(i) This is the "second-order form" of Chapter 15. Taking the cell state to be the pair (ϕ_j^n, ϕ_j^{n-1}), the recurrence is a one-step local update rule. Its inverse, $\phi^{n-1} = \cos\theta(\phi_{j+1}^n + \phi_{j-1}^n) - \phi^{n+1}$, has the same form. Running a wave packet 600 steps forward and 600 back returns the initial condition to 4.9×10^{-15} (K404). What Chapter 15 showed for Life and the ECA — "second-order form erases nothing" — appears here as another expression of the unitarity of the Dirac walk.

$$\boxed{\text{first-order, two-component, unitary} \iff \text{second-order, one-component, reversible}}$$

(ii) The continuum limit is the Klein–Gordon equation. With lattice spacing a and time step τ,

$$\frac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{\tau^2} = \frac{a^2\cos\theta}{\tau^2} \cdot \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{a^2} - \frac{2(1-\cos\theta)}{\tau^2}\, \phi_j^n .$$

Setting $a/\tau = c$ and $\theta = mc^2\tau/\hbar = mca/\hbar$ and taking $a, \tau \to 0$,

$$\boxed{\partial_t^2\phi - c^2\partial_x^2\phi + \left(\frac{mc^2}{\hbar}\right)^2 \phi = 0}$$

The angle that mixes left- and right-moving components becomes the mass in the continuum limit. After "mass is a ratio of lengths" (§6-1) and "mass is the angle joining the Compton length and the de Broglie wavelength" (§6-3), this is the third reading of the mass angle.

(iii) The lattice correction collects into one term. Expanding $\cos\omega = \cos k\cos\theta$,

$$\omega^2 = k^2 + \theta^2 - \tfrac{1}{3}\,k^2\theta^2 + O\big((|k|+|\theta|)^6\big).$$

Measuring $(\omega^2 - k^2 - \theta^2)/(k^2\theta^2)$ at $k=\theta=s$ gives $-0.3404,\ -0.3351,\ -0.3338,\ -0.3334$ for $s = 0.4, 0.2, 0.1, 0.05$, converging to $-1/3$ (K405). This $-k^2\theta^2/3$ is **the same term** as $v_\varphi v_g = 1-\theta^2/3$ of §6-5. The "lattice deviation" measured separately in two sections was one coefficient.

6-7-3. The limiting speed is $\cos\theta$ 🆕

The group velocity of $\omega(k) = \arccos(\cos k\cos\theta)$, $d\omega/dk = \cos\theta\sin k/\sin\omega$, attains its maximum at $k=\pi/2$ ($\omega=\pi/2$):

$$\boxed{v_{\max} = \cos\theta}$$

The third and fourth columns of the table in 6-7-1 show the numerical maximum agreeing with $\cos\theta$ to six digits for all five θ (K406). A wave packet ($k_0 = 0.6$, $\theta = 0.3$, width 40 sites, 600 steps) has centroid velocity 0.87717, agreeing with the dispersion group velocity 0.87701 to 1.8×10^{-4} (K407).

This adds a row to Chapter 24's classification. A species of mass angle θ has limiting speed $\cos\theta \simeq 1-\theta^2/2$, so **different species have different limiting speeds** — in form, the dimension-four violation of §24-2. But the coefficient is

$$1-\cos\theta \simeq \frac{\theta^2}{2} = \frac{1}{2}\Big(\frac{a}{\lambda_{\mathrm{C}}}\Big)^2,$$

which for an electron on a Planck lattice ($\theta \sim 4\times10^{-23}$) is 10^{-45}.

Chapter 24 class	Origin	Suppression	Verdict
Dimension 6 (24-1)	dispersion anisotropy	$(ka)^2/36$	harmless
Dimension 4 (24-2)	coupling differences **assigned by hand**, $v_i \propto t_i$	none	**the real problem**
Dimension 4′ (this section) 🆕	the $\cos\theta$ difference **generated automatically by the rule**	$(a/\lambda_{\mathrm{C}})^2$	harmless

Species dependence generated by the rule is suppressed by the square of the mass. Only species dependence put in by hand is unsuppressed.

The conclusion of §24-2 is unchanged; its scope is now precise.

6-7-4. Two limitations

Not finite-state. ϕ is complex. Making this a finite-state CA requires discretising the values, and the resulting error and continuum limit must be studied separately (fourth period). By the definition of Chapter 0, the rule of this section is not a "CA" but a "reversible linear map on a lattice".

Not a spin-0 quantum particle. That each component of the Dirac walk satisfies KG is the lattice version of the continuum fact that KG is the square of the Dirac equation; it does not derive a scalar quantum field. Bosonic commutation relations, quantum states, and measurement rules are the business of Part V. The restriction of §85-4 applies here too.

6-7-5. The form on a causal set 📚

On the irregular causal sets of Part XXII, with B_ℓ the operator approximating the flat-space $\Box$, the corresponding equation is

$$\Big[B_\ell - \Big(\frac{mc}{\hbar}\Big)^2\Big]\phi = 0.$$

In curved spacetime the Benincasa–Dowker operator approximates $\Box_g - \frac{1}{2}R$, so a minimally coupled scalar needs a curvature correction. B_ℓ refers arbitrarily far into the past inside the light cone (the side of Chapter 76's trilemma that gives up locality); it cannot be written with "one step back and the neighbours" as the recurrence of this section can.

Eight declarations, zero failures — but all declarations concern quantities with known analytic values, and are not semi-decisive in the sense of C-7.

6-8. Correction box — it is the dispersion relation, not de Broglie's relation, that breaks ⚠️

§6-5 wrote "on the lattice de Broglie's relation is not exact". **The attribution is inaccurate.**

Defining the crystal momentum $p = \hbar k/a$, $\lambda_{\mathrm{dB}} = 2\pi/k = h/p$ holds **exactly, by definition**. What the lattice breaks is $p = \gamma m u$, the relativistic relation among energy, momentum and velocity. The quantity measured in §6-5,

$$\frac{\gamma\theta u - k}{k} \simeq -\frac{k^2}{6} - \frac{\theta^2}{3},$$

is precisely the difference between k and $\gamma\theta u$ — a breaking of the dispersion relation. As seen in 6-7-2 (iii), this $-\theta^2/3$ is the same coefficient as the $-k^2\theta^2/3$ in ω^2; it belongs to "dispersion", not to "de Broglie".

	Tenth edition	Eleventh edition
Heading of §6-5	How much does the lattice break de Broglie's relation	How much does the lattice break the dispersion relation
Relation that breaks	$\lambda = h/p$	$p = \gamma m u$ (lattice correction to $E^2 = p^2 + m^2$)
Conclusion	deviation $\sim$ (mass)^2	**unchanged**

§6-6's "a CA does not 'derive' de Broglie's relation; it has it" also stands as written. What it has is $\lambda = h/p$, and that does not break. What breaks is the relativistic kinematics built on top of it. Recorded in Appendix C-10-2.

> ⚠️ **Correction (thirteenth edition).** The "**unchanged**" in the table above needs correcting. See the box of §6-5 and §87-5.

Chapter 7 Chirality and domain walls ✅

— Where does the distinction between right- and left-handed come from

Chiral fermions live on a domain wall (Jackiw–Rebbi, Callan–Harvey).

Test	Result
slopes of left/right dispersions	∓ 0.99179
sum (chirality cancellation)	3.3×10^{-16}

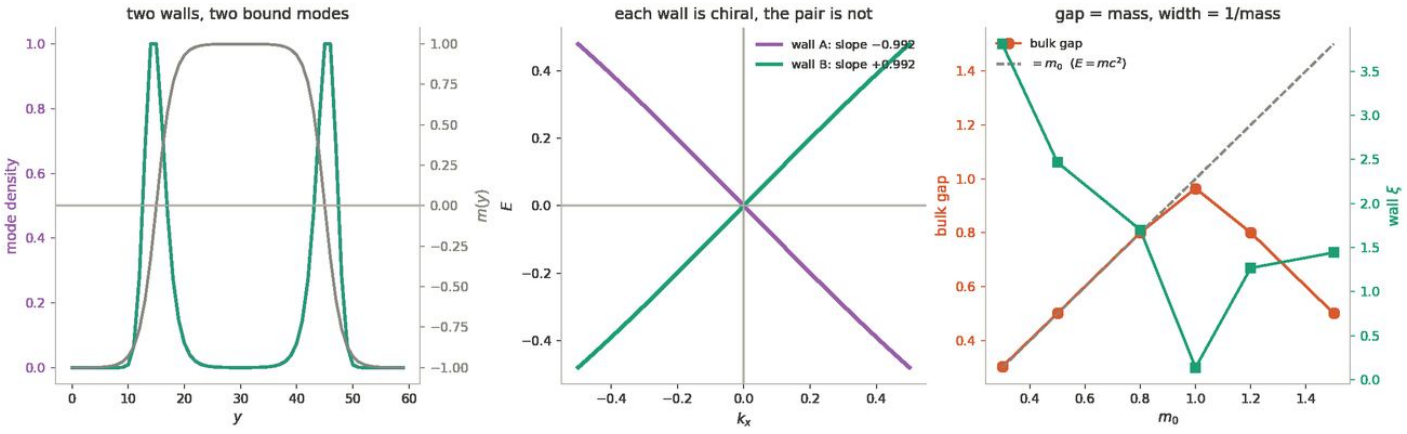


Figure 7-1. Chiral modes bound to a domain wall; the slopes cancel to 10^{-16}.

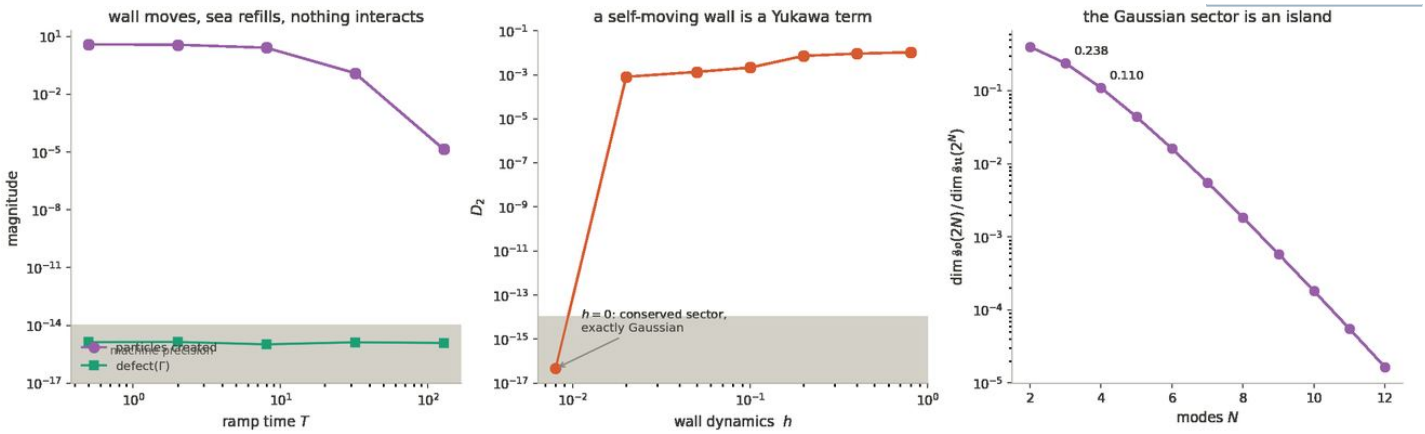


Figure 7-2. Moving the wall. This becomes the main tool when Part VIII discusses selection of the mass ratio.

Chapter 8 Exclusion ✅ / but it depends on dimension ⚠️

— Why can two electrons not occupy the same state

A two-state cell is hard-core statistics. Only 0 or 1 fits in a cell, so exclusion is not a rule but the definition of the state space. Matter also fails to collapse (E/n from -4.04 to -3.33, Pauli pressure).

But the dimension dependence is sharp.

Dimension	Statistics	Mechanism
$d = 1$	fermions	Jordan–Wigner is exact
$d = 2$	**anyons**	exchange phase arbitrary; JW becomes non-local
$d = 3$	fermions/bosons	the phase group is $\mathbb{Z}_2$

"Two-state cell = fermion" holds only in one dimension.

Changing the fermionisation convention destroyed Gaussianity by **15 orders of magnitude**. What is dimension-specific is not the content of a cell but the **bandwidth** — how non-local a string must be dragged along.

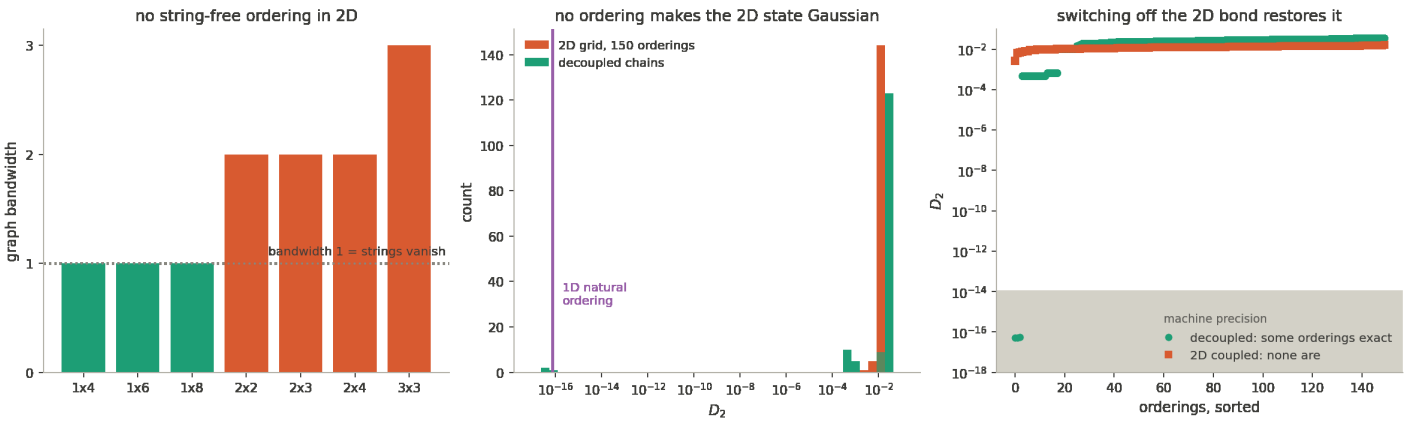


Figure 8-1. Fermionisation in two dimensions: locality cannot be maintained.

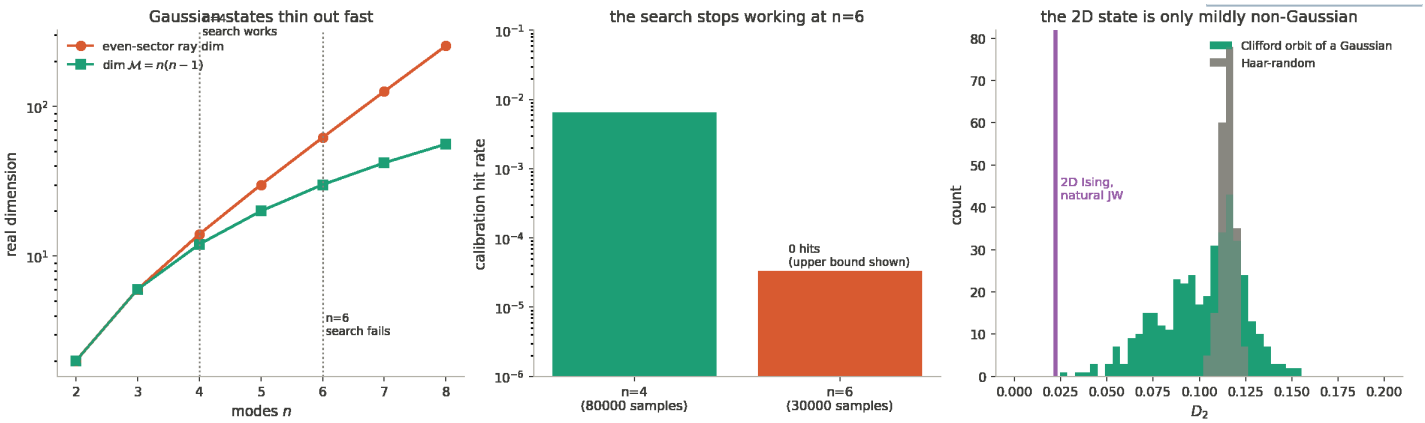


Figure 8-2. Changing convention destroys Gaussianity by 15 digits.

> 💭 In real physics the spin-statistics theorem is a theorem of relativistic field theory. A CA has no relativity, so statistics is **not derived but an input that changes with dimension**. It enters the checklist of Part X.
>
> ⚠️ **Restriction (thirteenth edition).** A two-state cell gives a limit on local occupation. Anticommutation relations and the antisymmetry of many-particle states are further input (hard-core bosons are possible). "Exclusion" in the heading is to be read in this sense. §87-9.

Chapter 9 Summary of Part II — how far was the wish granted

— What came for free, and what came with a price tag

Element	Origin in a CA
causality	the rule sees only the neighbourhood
finiteness	no ultraviolet divergence
uncertainty relation	the definition of $\mathcal{M}$
zero-point motion	its saturation condition
vacuum / particle / Dirac sea	one and the same geometric object
$E = mc^2$	mass is a ratio of lengths
chiral fermions	bound to a domain wall
exclusion	the two-state cell ($d = 1$)
metric, expansion, holonomy	space- and time-dependence of the rule

Nothing here was put in by hand. Writing down a simple local rule already yields all of it.

Up to here the wish is granted. The walls begin in the next chapter.

> ⚠️ **Restriction (thirteenth edition).** "Nothing here was put in by hand" is to be read under the premise of §85-4 (the triple of Chapter 2) and the restrictions of §87-9. "Metric" means an effective causal structure.

Part III The first wall — geometry does not generate interaction

Everything gained in Part II was a property of **free fields**. Here we hit the wall. And find the way past it. **This is the most affirmative part of the book.**

Chapter 10 Solvable models form an island of measure zero ✅

— Why are exactly solvable models so rare

10-1. Bilinears close

The commutator of Majorana bilinears is again a bilinear (numerical residual 10^{-15}). This is the fact that $\mathfrak{so}(2N) \subset \mathfrak{su}(2^N)$.

10-2. How small is the island

$$\frac{\dim \mathfrak{so}(2N)}{\dim \mathfrak{su}(2^N)} = \frac{N(2N-1)}{4^N - 1}$$

N	3	6	12
ratio	0.238	1.6×10^{-2}	10^{-6}

Solvable models are an island of measure zero among possible dynamics.

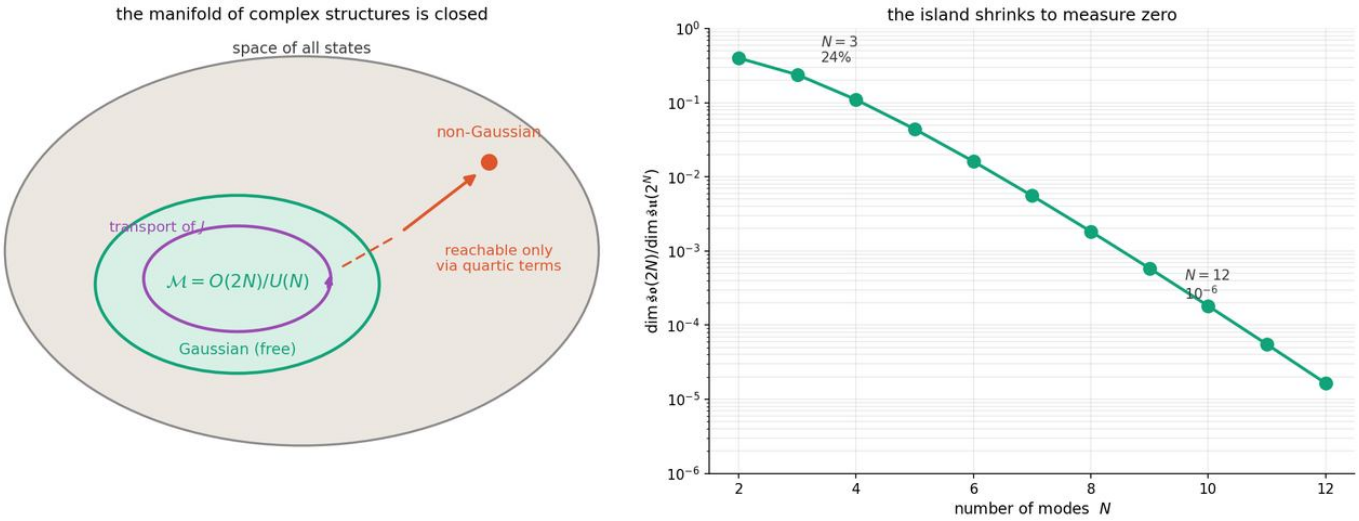


Figure 10-1. The island $\mathcal{M}$ and the non-Gaussian sea around it.

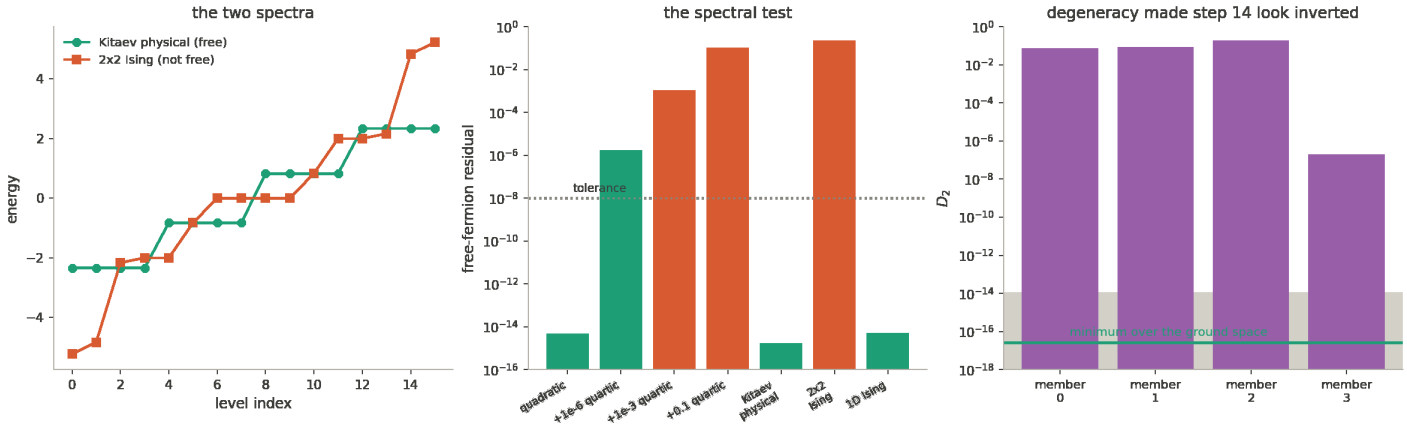


Figure 10-2. Diagnosing solvability from the spectrum.

Chapter 11 $D_2 \sim 10^{-32}$ — not a failure but a theorem ✅

— Why does adding interaction to a free theory break it

11-1. Record of the search

To leave $\mathcal{M}$ we tried all of the following: transporting J with space dependence; with time dependence; taking holonomy around closed loops; adding static gauge fields; adding mass terms.

All failed. The non-Gaussianity indicator stayed at

$$D_2 \sim 10^{-32}.$$

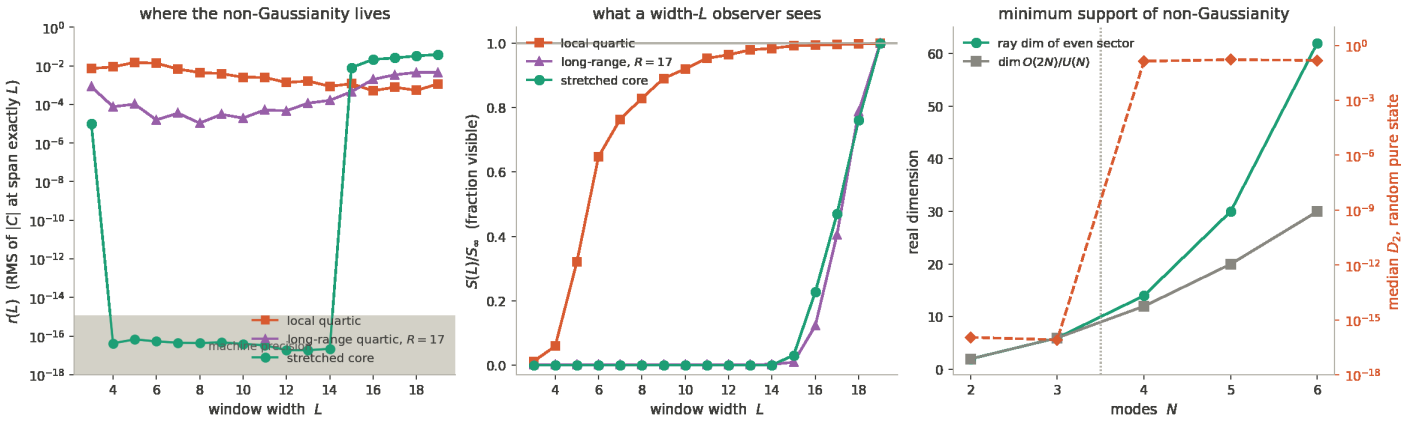


Figure 11-1. Windowed non-Gaussianity. No operation moves it off 10^{-32}.

11-2. Why — one line in the algebraic language

Any motion on $\mathcal{M}$ is an $O(2N)$ action and preserves $\Gamma^2 = -\mathbb{1}$. Algebraically:

$$\boxed{\text{Bogoliubov automorphisms map quasi-free states to quasi-free states.}}$$

$$\boxed{\text{Geometry does not generate interaction.}}$$

> 💭 **In the algebraic language this is closer to a definition than a theorem.** Then why did it take forty-one experiments? **Because the numbers came first and the language second.** Only after knowing where the search fails does one see that a word is needed for why. As a textbook, that order must not be erased.

In real physics the geometry of spacetime acts on matter. On $\mathcal{M}$, however you move the geometry, matter stays free. **Interaction must come from somewhere else.**

Chapter 12 Constraints force interaction ⚠️ (restricted in the thirteenth edition; §87-6)

— Where does interaction come from

The way past the wall. The most affirmative result in the book.

12-1. Contrast with reality

In real physics interaction terms (ϕ^4, $\bar{\psi}\gamma^\mu\psi A_\mu$) are **written into** the Lagrangian. Symmetry restricts them, but not writing them is an option.

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12-2. In this model they cannot be written as bilinears ✅ (heading restricted in the thirteenth edition)

A gauge-invariant hopping $\tau^z\gamma\gamma$ has **residual** 1.00 from the span of bilinears.

Condition	Non-Gaussianity D_2
gauge invariant (local conservation law)	3.07×10^{-2}
gauge invariance dropped	9.08×10^{-17}

Fifteen orders of magnitude.

Quartic terms are not added by hand. They are a consequence of local conservation laws.

⚠️ **Restriction (thirteenth edition).** The comparison in this box is within this model. "A consequence of local conservation laws" is to be read as "in this model, with a Gauss-type constraint and an electric term". §87-6.

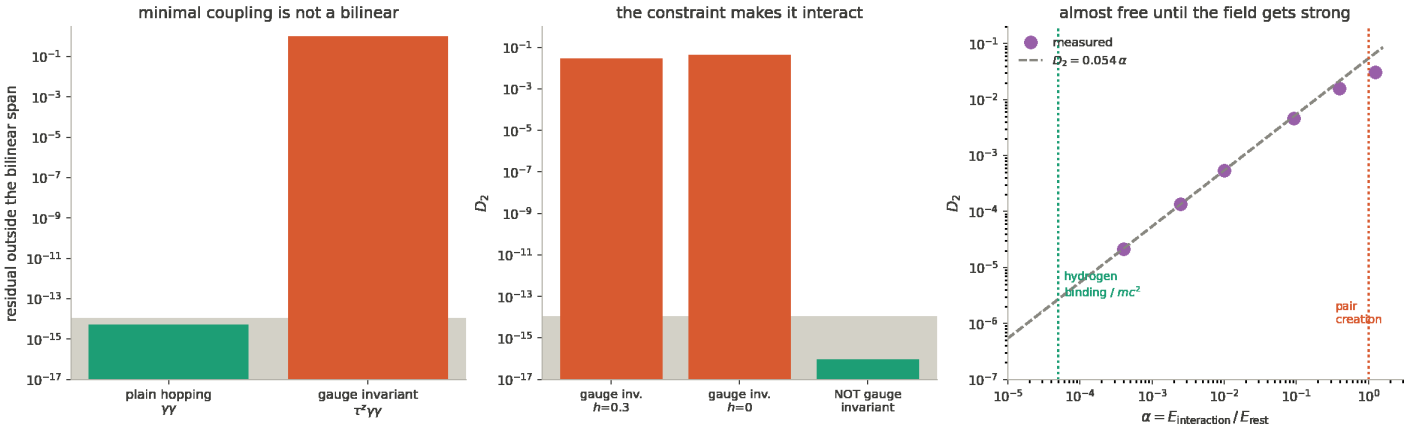


Figure 12-1. Imposing the Gauss law forces terms that no bilinear can write.

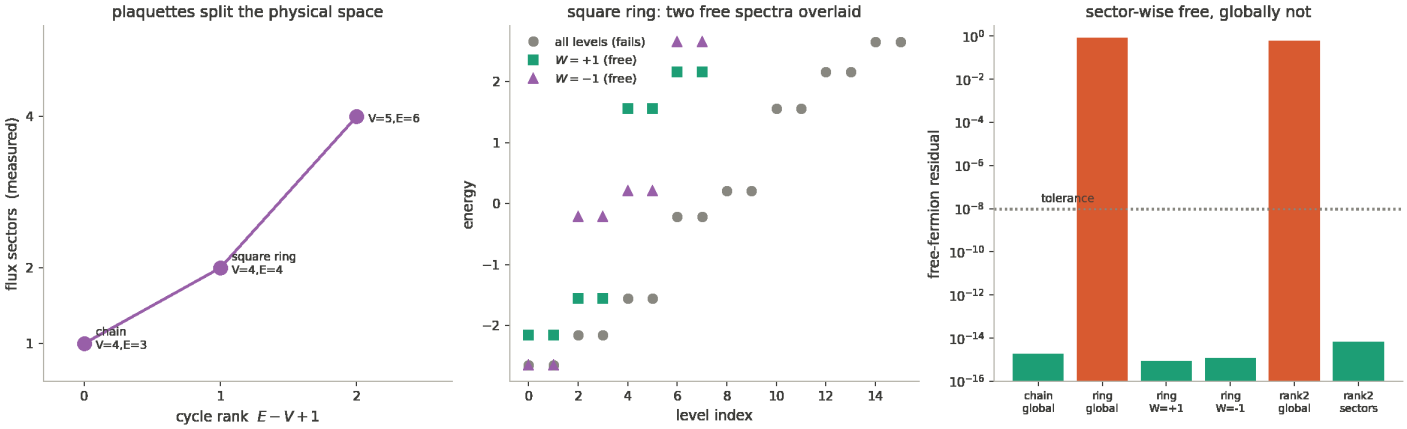


Figure 12-2. Plaquette terms and gauge-field structure.

Quantitatively,

$$D_2 \simeq 0.054\,\alpha, \qquad \alpha = \frac{E_{\text{interaction}}}{E_{\text{rest}}}$$

💭 **For the same reason that ordinary light looks linear, ordinary matter is describable by an almost-free CA.** The nonlinearity has not vanished; α is merely small.

12-3. New theorem ✅

$$\boxed{\text{The number of inter-site bilinears commuting with a local constraint is } \mathbf{zero}.}$$

Under a constraint, any non-trivial dynamics **must** contain quartic or higher terms.

$$\boxed{\text{The tension is not "CA vs physics" but "solvability vs interaction".}}$$

> ⚠️ **Restriction (thirteenth edition).** The theorem concerns the inter-site bilinears of this model. A one-dimensional $\mathbb{Z}_2$ model with no electric term, under the Gauss law, has the eigenvalues of the free chain (K433). What carries interaction is a Gauss-type constraint together with the electric term. §87-6.

Chapter 13 The chain of consequences — Mott, Cooper, hierarchy ✅

— Mott insulators, Cooper pairs, mass hierarchy — one mechanism or three

Impose one constraint and physics follows in a chain.

13-1. On-site U ✅

Schrieffer–Wolff perturbation theory gives $U = 0$ to all orders (1.1×10^{-13}). **But that was without imposing the Gauss law.** With the constraint the electric field is enslaved to the charge and

$$\boxed{U = 2h\left[1 - \cos(2\pi/N)\right] > 0 \qquad \forall N \geq 2}$$

$\mathbb{Z}_2$ gives the largest ($4h$). **The simplest group gives the strongest interaction.**

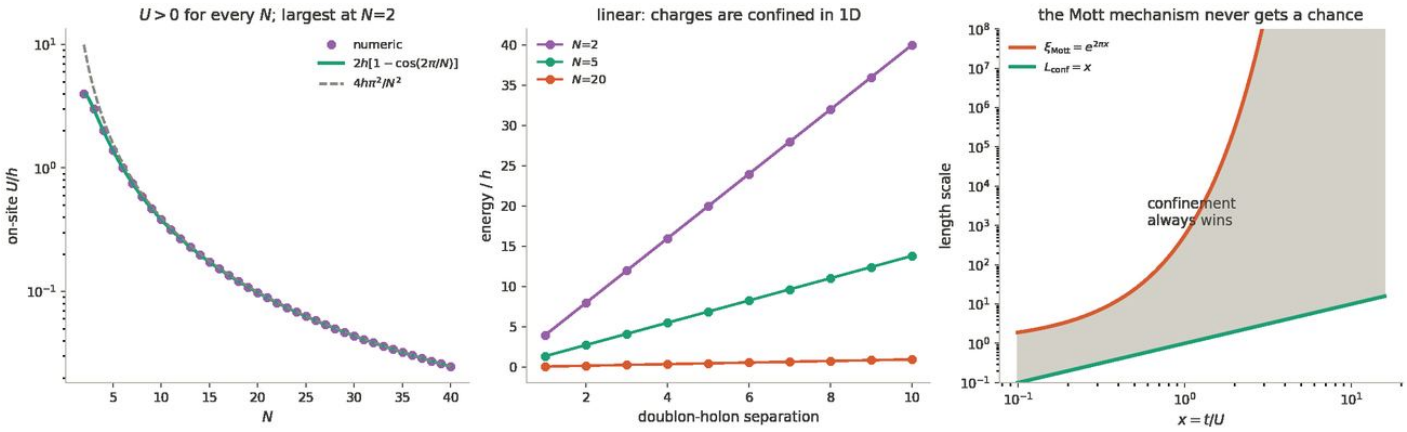


Figure 13-1. The electric term generates an on-site interaction fixed by the group $\mathbb{Z}_N$.

> This formula plays a decisive role in Part VIII, because **the coupling becomes a lattice of points labelled by integers rather than a continuous dial**.

13-2. Deconfined Mott phase ✅

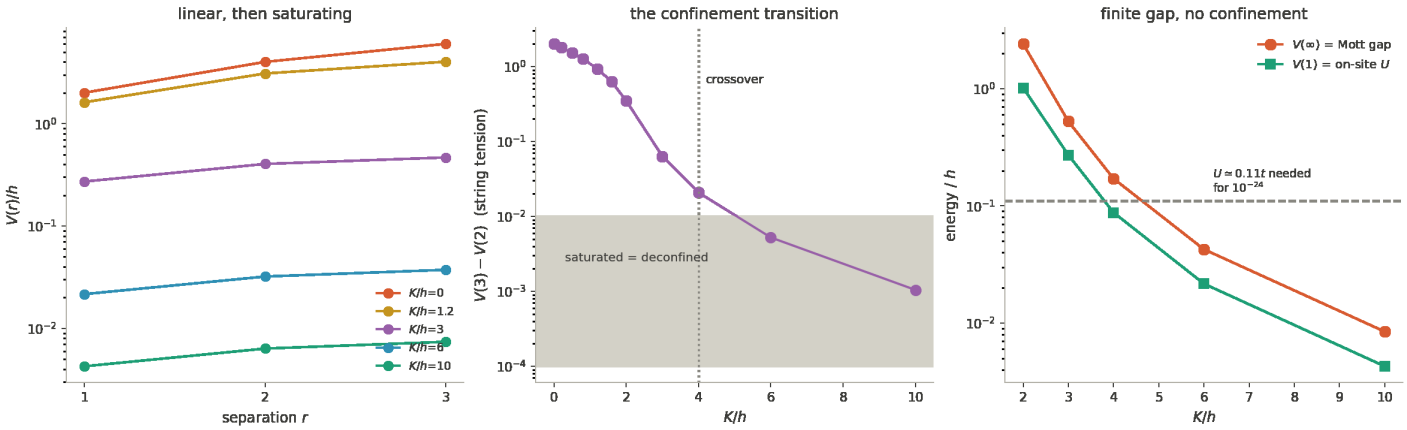


Figure 13-2. For $d \geq 2$ there is a deconfined phase with a finite Mott gap.

13-3. Cooper pairs come out of confinement ✅

No attraction was assumed. As a consequence of confinement, $E_b < 0$ and the pair bandwidth is $3.8\times$ that of a single particle.

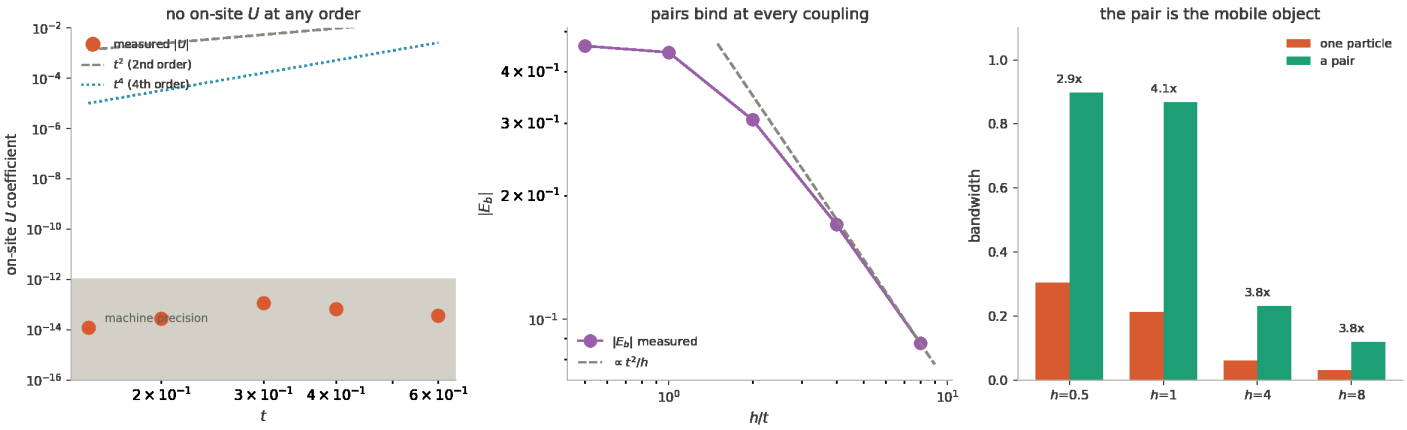


Figure 13-3. A bound pair. The binding is power-law, a different mechanism from BCS.

13-4. The exponential hierarchy is not fine tuning ✅

The CA interaction is marginal ($\langle nn\rangle_c \sim r^{-2.0386}$, $R^2 = 0.99992$); irrelevant for spinless, **relevant for spinful** — spin halves the dimension of the umklapp operator.

$$\Delta_c \sim e^{-2\pi t/U}, \qquad \frac{\ln(1/\Delta_c)}{2\pi/U} \to 0.968$$

$U \approx 0.11\,t$ gives 10^{-24}. For 34 orders of magnitude, U moves by a factor 2.4.

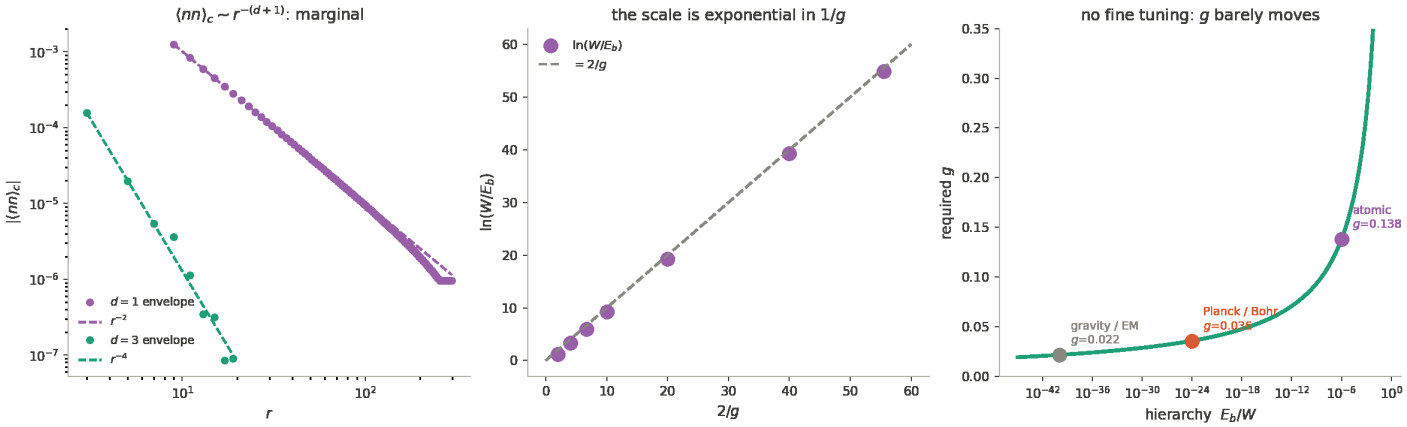


Figure 13-4. Dimensional transmutation: exponential amplification makes the hierarchy.

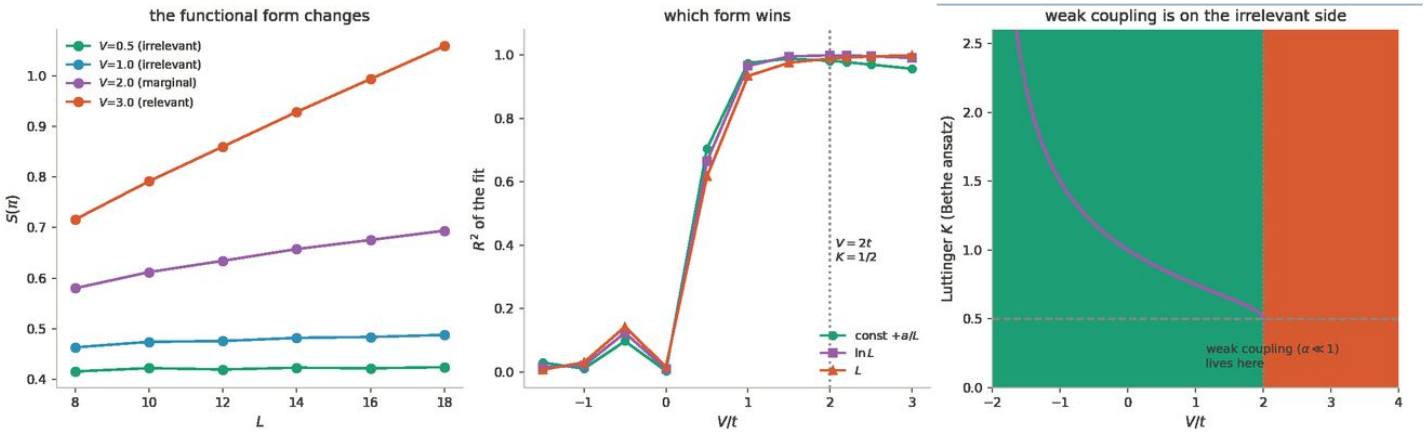


Figure 13-5. Marginal flow.

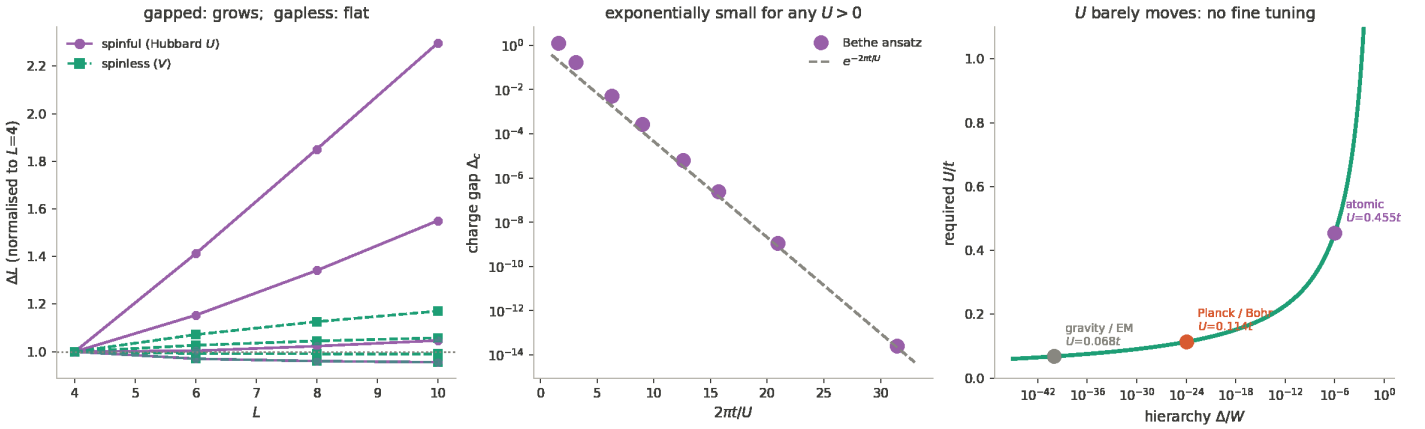


Figure 13-6. Spin halves the umklapp dimension.

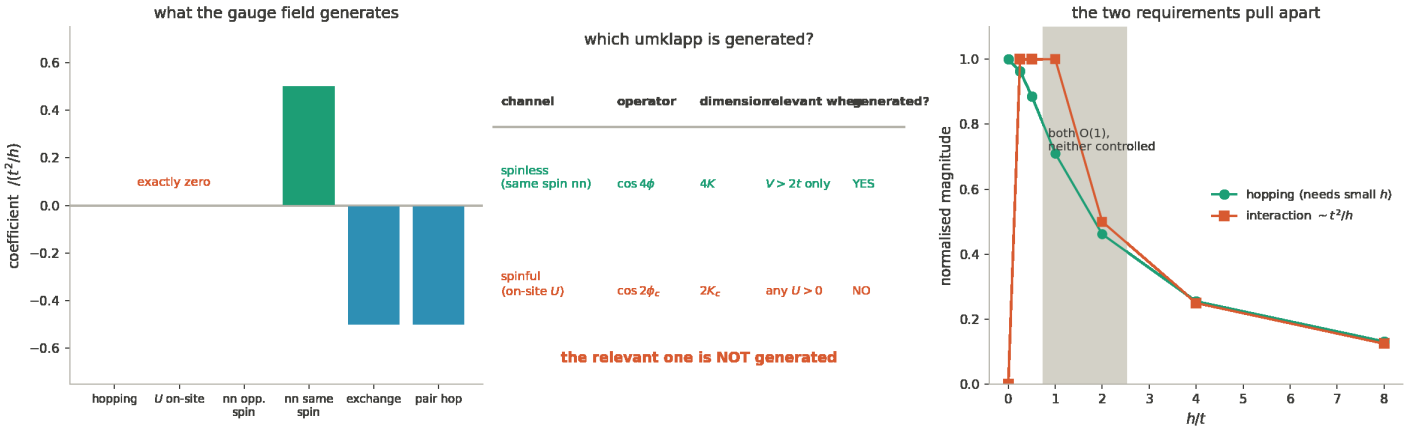


Figure 13-7. Direct check of the umklapp dimension: $\langle nn \rangle_c \sim r^{-2.0386}$, $R^2 = 0.99992$.

The requirements are: at least two states per cell, commensurate filling, repulsion. **Cheap for a CA.**

Reality: interactions are written into the Lagrangian. CA: impose a local conservation law and interactions are **forced**.

This is one of the few places where a CA explains more than reality does. Asked why interactions exist, it answers: because there are constraints.

> ⚠ **Restriction (thirteenth edition).** "Forced" and "explains more" in the box above are withdrawn. The formula of §13-1 itself gives $U = 0$ at $h \to 0$. §87-6.

Part IV Time — heat, entropy, arrow

Chapter 14 $k_B = 1$

— What does temperature actually measure

A CA has no independent unit of temperature; it can only count states, so $k_B = 1$ by construction. **That in itself is bookkeeping, not physics.**

What is interesting is the following three, all structural.

Item	Reality	CA
Landauer bound	$k_B T \ln 2$ must be dissipated	**exactly zero if reversible**
third law	entropy tends to a constant as $T \to 0$	**there is no bottom to fall to**
arrow of time	second law	**the law has no direction**

> ⚠️ **Addition (twelfth edition).** "The law has no direction" stands. Planck's inequality too is time-symmetric: it holds for both U and U^{-1} and only imposes a condition on the initial state. It does not explain a low-entropy past. §86-12.

Chapter 15 The Landauer bound disappears ⚠️🆕 (reading of the numbers corrected in the thirteenth edition; §87-4)

— Why does erasing information cost heat

15-1. Here the book runs Life for the first time

The forty-one first-generation experiments never once ran the Game of Life. It is run here **to measure Life's thermodynamic price**.

In a CA the number of erased bits can be counted exactly: enumerate the whole state space and count the image.

$$\Delta S_{\text{bit}} = \log_2 |\Sigma^{\mathcal{L}}| - \log_2 |F(\Sigma^{\mathcal{L}})|$$

Rule	System	States	Image	Erased/step	Per cell
Conway Life	4×4 torus	65 536	17 879	1.8740 bit	**0.1171**
Conway Life	5×4 torus	1 048 576	279 165	1.9092 bit	**0.0955**
ECA 110	ring $L = 18$	—	—	1.3753	0.0764
ECA 90	ring $L = 18$	—	—	**2.0000**	0.1111
ECA 150	ring $L = 18$	—	—	**2.0000**	0.1111
ECA 30	ring $L = 18$	—	—	0.0374	0.0021
2nd-order 110	ring $L = 9$	262 144	262 144	**0**	**0**
2nd-order 30	ring $L = 9$	262 144	262 144	**0**	**0**

Three readings.

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(a) Linear rules erase an integer number of bits. ECA 90 and 150 erase exactly 2 — the kernel of a linear map over $\mathbb{F}_2$ has dimension exactly 2. Algebra, not rounding.

(b) Life discards about 0.1 bit per cell per step. At 300 K,

$$0.0955 \times 2.871 \times 10^{-21}\ \mathrm{J} = 2.74 \times 10^{-22}\ \mathrm{J\ per\ cell\ per\ step}.$$

(c) The second-order form gives exactly zero. The rule was not changed. **Only the definition of the state was.**

$$\boxed{\text{A reversible CA universe emits no heat, however much it computes.}}$$

> 💭 The Landauer bound is not a property of the universe but **the price of cutting a piece out of it and calling it a device**. Erasure comes from the definition of the state — what is remembered and what is thrown away — not from the rule.
>
> ⚠️ **Correction (thirteenth edition).** The "erasure" of the table above is a quantity of image size, a lower bound on the uniform-input loss $H(X) - H(F(X))$. The uniform-input loss per cell is 0.3070 for Life 4×4, 0.2615 for 5×4, 0.1078 for ECA 110, 0.0028 for ECA 30 (K431). The second-order form is a different dynamics (K432), and what vanishes is logical erasure. Heat needs a bath and an implementation. §87-4.

Chapter 16 Fine-grained vs coarse-grained — the arrow is in the partition 🆕

— Where exactly does the arrow of time live

Quench a reversible CA and measure both entropies. $n = 200$ Majoranas, subregion of 20 sites.

τ	S_A (coarse)	S_{total} (fine)	$\lVert\Gamma^2 + \mathbb{1}\rVert/n$
0	0.2701	4.8×10^{-12}	1.8×10^{-15}
2	0.9483	4.5×10^{-12}	1.7×10^{-15}
8	1.9676	4.1×10^{-12}	1.6×10^{-15}
32	1.9802	4.0×10^{-12}	1.6×10^{-15}
256	1.6834	3.9×10^{-12}	1.6×10^{-15}

$$\boxed{S_A : 0.27 \to 1.98\ (\times 7), \qquad S_{\mathrm{total}} : \text{fixed at } 10^{-12}}$$

The dip at $\tau = 256$ is a finite-size revival. **Entropy is not even monotonic.**

$$\boxed{\text{The arrow of time is not in the law. It is in the partition.}}$$

This agrees **completely** with real physics. What differs is only that a CA makes it impossible to hide.

16-1. The long-time fate

Free fermions are integrable and do not thermalise; they relax to a generalised Gibbs ensemble.

Quench	Plateau	Ground state	Residue
$\mu : 2.2 \to 1.0$	0.20505	0.17858	$+0.02647$
$\mu : 1.6 \to 1.0$	0.19393	0.17961	$+0.01431$
$\mu : 1.2 \to 1.0$	0.18472	0.17858	$+0.00614$

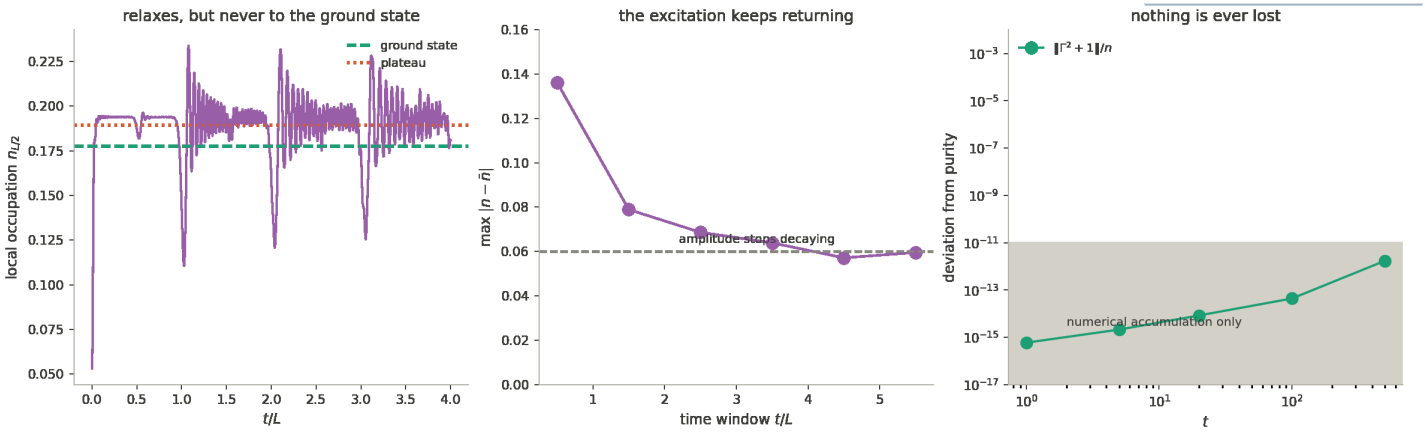


Figure 16-1. Long-time evolution. Purity is exactly 1 even at $t = 500$; the oscillation stops falling at $0.136 \to 0.060$.

> There is no collapse and no information loss. There is only redistribution.

Chapter 17 The third law fails — there is no bottom 🆕

— Why can absolute zero not be reached

17-1. The premise in reality

$$\lim_{T \to 0} S(T) = S_0$$

Nernst's theorem presumes **a ground state**, i.e. a spectrum bounded below.

17-2. In a CA quasi-energies live on a circle

$$U|\psi\rangle = e^{-i\varepsilon}|\psi\rangle, \qquad \varepsilon \in (-\pi, \pi]$$

ε and $\varepsilon + 2\pi$ are indistinguishable. **The spectrum lies on a circle, not on a line.**

Confirmed by two independent calculations.

(a) Floquet free-fermion circuit

θ	spectral width	largest empty arc
0.2	0.800	5.483
1.2	4.800	1.483
$\pi/2$	$\mathbf{6.283 = 2\pi}$	$\mathbf{3.1 \times 10^{-3}}$

(b) A genuine reversible CA (second-order rule 110; eigenvalues of a permutation matrix are roots of unity)

L	states	cycles	cycle length min/max	distinct quasi-energies	largest empty arc
6	4 096	126	1 / 90	613	0.0698
8	65 536	500	1 / 940	9 701	**0.0067**

The gap keeps shrinking with system size.

> The quasi-energies of a reversible CA fill the unit circle densely. The generator that would define a "lowest" is not uniquely fixed by U.

⚠️ **Wording changed in the eleventh edition.** The tenth edition wrote "there is no 'lowest'". The precise form, as §17-3 already has it, is that a local, bounded-below H is not uniquely fixed by U. §85-5.

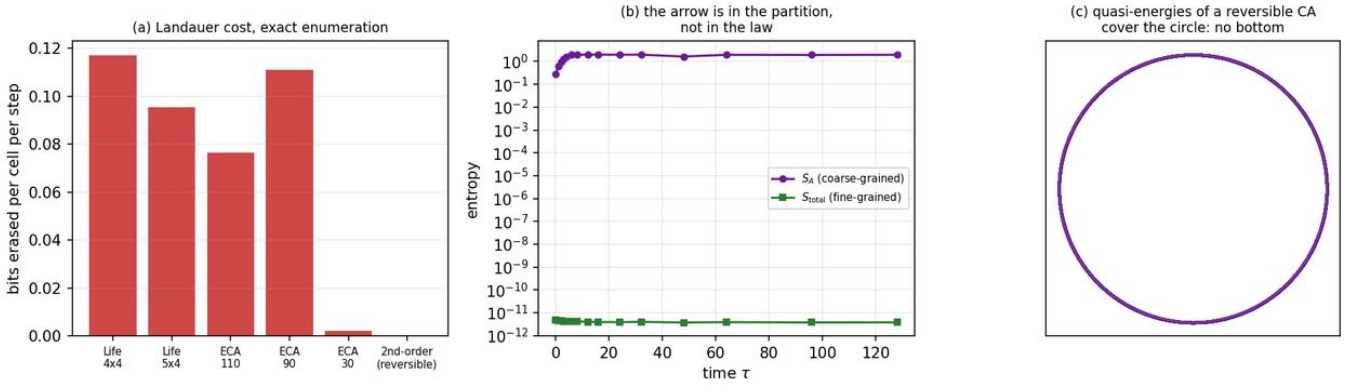


Figure 17-1. 🆕 (a) Bits erased per step (exact enumeration); red irreversible, green reversible. (b) Coarse-grained (purple) and fine-grained (green) entropy after a quench. (c) Quasi-energies of a reversible CA covering the circle ($L = 8,9701$ points).

17-3. Getting the attribution right 💭

"There is no ground state" is a correct conclusion but the cause is misattributed. In the algebraic language:

Discrete time means the evolution is generated by a $\mathbb{Z}$-action, not $\mathbb{R}$. The dual of a $\mathbb{Z}$-action is a circle, and a circle carries no order structure. One cannot even write "$H \geq 0$" — **the inequality is undefined**.

> Neither the algebra nor the automorphism is broken.
> **The predicate "spectral condition" does not exist in the language.**

This has a substantive consequence. If the algebra and the automorphism are sound, embedding $\mathbb{Z}$ into $\mathbb{R}$ (supplying a continuous-time generator) can repair it. **It is not broken; it has not yet been written.**

💭 Of the three assumptions, the most expensive physically is **discrete time**. Discrete space breaks Lorentz invariance; discrete time breaks energy itself.

⚠️ **Correction (thirteenth edition).** "The inequality cannot be defined" is wrong. In a finite system $H \geq 0$ can be chosen for any unitary (K434). The correct form is "a local, bounded-below H is not uniquely fixed by U". §87-7.

Part V Quantumness — where the classical CA dies

Chapter 18 Superposition cannot be built ❌

— Can superposition be built out of a deterministic rule

The state space of a classical discrete-time CA is a **set**. An element of $\Sigma^{\mathcal{L}}$ says which cell has which value; superposition is undefined.

A reversible CA can be lifted to a **permutation matrix** on a $2^{|\mathcal{L}|}$-dimensional Hilbert space. But:

$$\boxed{\text{A permutation matrix never generates entanglement from a product state.}}$$

From $|0\cdots0\rangle$ one never reaches $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

There is no way around this. If superposition is needed, the rule must be unitary rather than a permutation.

$$\text{classical CA} \longrightarrow \text{quantum cellular automaton (QCA)}$$

As seen in Chapter 2, **both sides of this fork are CA**, because CHL and GNVW are the commutative and non-commutative versions of one theorem.

Chapter 19 Measurement — three ways out of $\mathcal{M}$

— What actually happens when the wave function collapses

A reversible CA has no projection: there is one rule and it is reversible.

Numerically, **projection destroys Gaussianity**. A state Gaussian in an enlarged space becomes non-Gaussian when projected to the physical space.

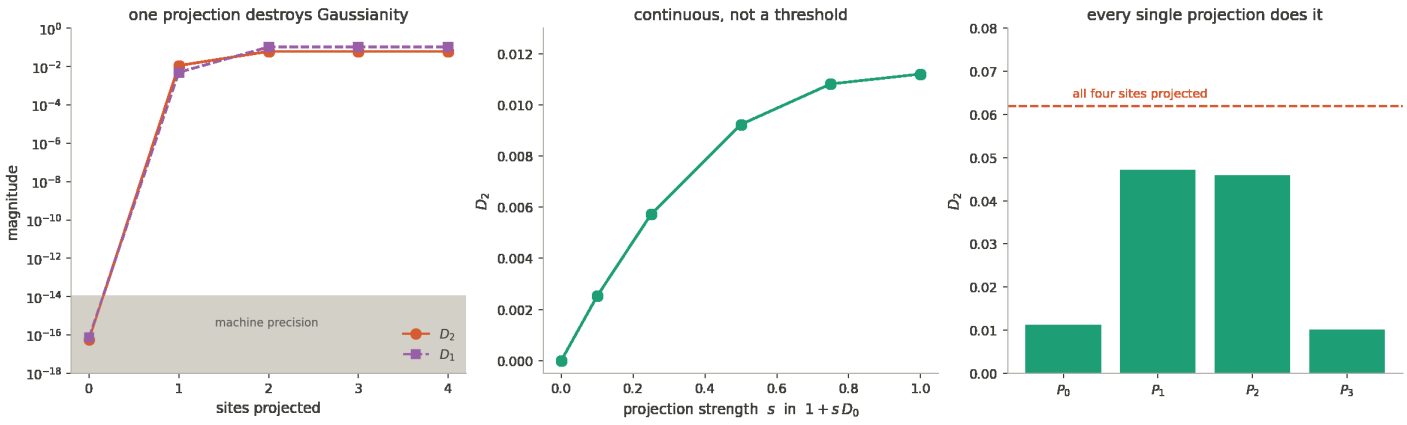


Figure 19-1. Projection destroys Gaussianity.

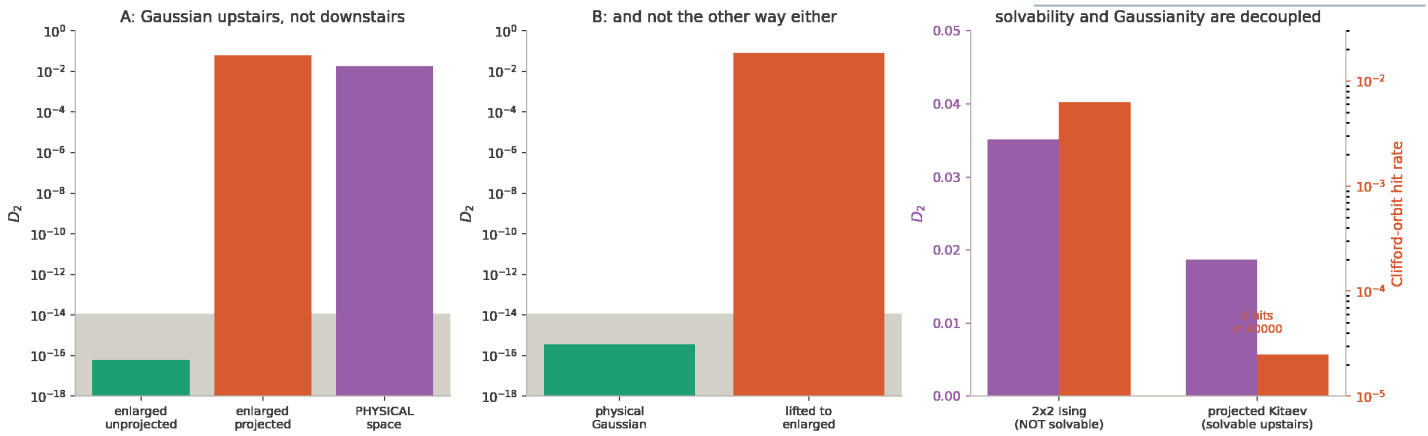


Figure 19-2. Projection from the enlarged space to the physical space.

$$\text{Three ways out of } \mathcal{M}: \quad (1)\ \text{local conservation laws} \quad (2)\ \text{quartic terms} \quad (3)\ \text{projection}$$

Algebraically, each breaks a different slot.

Route	Slot broken
conservation law → quartic term	α ceases to be Bogoliubov
projection (measurement)	ω ceases to be quasi-free
change of algebra (refermionisation)	the local structure of $\mathcal{A}$

There are exactly three ways to treat measurement in a CA universe.

Position	Content	Price
Everett	no projection, only branching	globally deterministic; probability needs a separate account
objective collapse	add a stochastic term to the rule	no longer reversible
superdeterminism	settings are fixed by the rule too	measurement independence abandoned

Taking the third means facing the next chapter head-on.

Chapter 20 Bell's theorem — what is being tested is not determinism 🆕

— What is Bell's inequality actually testing

A monograph that ducks this is not to be trusted.

20-1. Where the bound $\mathcal{S} \leq 2$ comes from

$$\boxed{\text{commutative algebra} \xrightarrow{\text{Gelfand}} \text{a joint distribution exists} \xrightarrow{\pm 1\text{-valued}} \mathcal{S} \leq 2}$$

Step one: a commutative C^*-algebra is an algebra of functions on a compact space, so all observables are simultaneously realised on one probability space.

Step two: for four ± 1-valued variables with a joint distribution, at each sample point $A_1(B_1 + B_2) + A_2(B_1 - B_2) = \pm 2$.

Step three: existence of a local hidden-variable model and existence of a joint distribution are equivalent (📚 Fine 1982).

$$\boxed{\text{Nowhere in this chain does "determinism" appear.}}$$

20-2. Hence what fails is commutativity ❌

What fails	Why
classical **deterministic** CA	the accessible algebra is commutative
classical **stochastic** CA	**also commutative; also fails**

The second line matters. Putting randomness into the rule leaves the algebra commutative as long as observables are functions on configuration space, and the bound is unchanged.

> Adding indeterminism does not break Bell. Non-commutativity does.

Moreover, by Koopman–von Neumann a classical reversible CA is also a **unitary** on ℓ^2. The contrast "classical = deterministic / QCA = unitary" is wrong: **both are unitary and both are deterministic.** Only the accessible algebra differs.

> ⚠️ **Restriction (thirteenth edition).** "Non-commutativity is what violates" is a statement within the frame where the four observables live on one commutative algebra. Models that drop premises such as locality and setting independence need separate treatment. §87-9.

20-3. Which part of non-commutativity is the resource 🆕

Two kinds of commutativity appear in CHSH, with opposite roles.

Commutativity	Requirement	Role
across the cut $[\mathcal{A}_A, \mathcal{A}_B] = 0$	**necessary**	Einstein causality; without it there is no Bell test
same side $[A_1, A_2] = 0$	**must not hold**	this is the resource

The reason lies in continuous settings. For $O(t) = \cos t\, X + \sin t\, Y$,

$$O(t)^2 = \mathbb{1} + \cos t \sin t \,\{X, Y\}$$

X, Y	$\{X, Y\}$	Is $O(t)$ dichotomic?
anticommuting	0	✅ for every t
commuting	$2XY$	❌ defect $= \sin 2t$ (1.0 at $t = \pi/4$)

> Commuting observables cannot form a setting tilted by 45 degrees; settings freeze onto discrete points.

The optimal CHSH settings are mutually at 45 degrees. Whether they can be built is what separates 2 from $2\sqrt{2}$.

Tested on the identical state, algebra and evolution.

Family of same-side settings	max CHSH
commuting family (exhaustive search)	**2.000000**
anticommuting triple (continuous rotation)	**2.828427**

20-4. Calibration (six items)

The observables are $\Sigma^A = (i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2)$ with $\Sigma_1\Sigma_2\Sigma_3 = -i\mathbb{1}$.

Test	Residual
$\|\Sigma^2 - \mathbb{1}\|$	0.00
$\|\{\Sigma_i, \Sigma_j\}\|$ (same side)	0.00
$\|[\Sigma^A, \Sigma^B]\|$ (**Einstein causality**)	0.00
$\|C_{\text{exact}} - C_{\text{Wick}}\|$	2.2×10^{-16}
Horodecki vs direct maximisation	agree
Tsirelson bound exceeded	no

> 💭 The third was included as a check that the observables really separate. Only after rewriting in algebraic language did it become clear that this was **numerically verifying the second Haag–Kastler axiom**.

20-5. Result — two slots are needed 🆕

(a) The algebra slot: shared Majorana pairs across the cut, count k, pairing angle θ.

k	$\theta = \pi/2$	$\theta = \pi/3$	$\theta = \pi/6$
1	2.000000	2.000000	2.000000
2	**2.828427**	2.121320	2.000000
3	**2.828427**	2.121320	1.581139

With one shared pair the connected Wick terms vanish, the correlation matrix has rank 1, and $\mathcal{S}$ freezes at 2. **The correlation from the smallest unit of fermionic entanglement is exactly reproducible by a commutative model.** Two pairs saturate Tsirelson.

(b) The state slot: ground states do not violate. Critical chain (with local frames optimised) reaches 1.993; the two-spin Ising case bottoms out at 1.681 at criticality. **Monogamy** is at work.

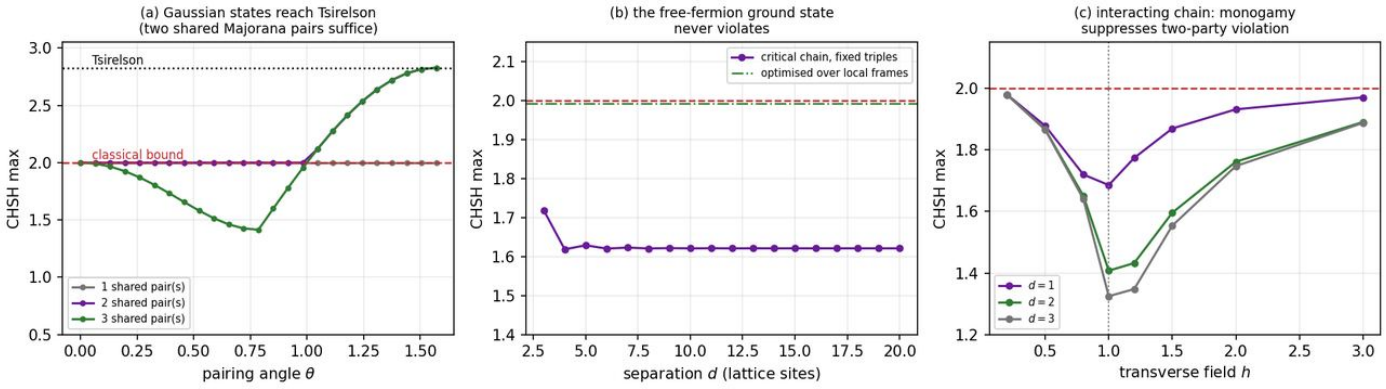


Figure 20-1. 🆕 (a) Two shared pairs reach Tsirelson. (b) The critical-chain ground state never exceeds the classical bound even with optimised local frames. (c) Same for an interacting chain; the minimum at criticality is the fingerprint of monogamy.

20-6. Consequence for the wish 💭

$$\begin{aligned} \text{algebra } \mathcal{A}: &\ \text{same-side settings must be } \textbf{non-commuting} \\ \text{state } \omega: &\ \text{entanglement must be } \textbf{concentrated} \text{ in the pair} \\ \text{automorphism } \alpha: &\ \textbf{irrelevant} \end{aligned}$$

Simplicity need not go. What must go is **commutativity**.
Indeterminism **comes along** as a consequence.

The direction of the logic matters. Indeterminism is not a requirement but a **consequence**.

$$\text{non-commutative algebra} \to \text{incompatible observables} \to \text{Born rule} \to \text{indeterminism}$$

And simplicity is untouched: $M_d(\mathbb{C})$ is no longer to write down than $\mathbb{C}^d$.

> ⚠️ **Restriction (thirteenth edition).** "Non-commutative algebra → Born rule → indeterminism" is not an implication. The Born rule does not follow from non-commutativity alone and, as in §75-8, has not been derived. §87-9.

Part VI Fields — electromagnetism

Chapter 21 The Bianchi identity is exact on a lattice ✅🆕

— Where does the law forbidding magnetic monopoles come from

In differential forms, Maxwell is two equations:

$$d \star F = \star J \quad \text{(law)}, \qquad dF = 0 \quad \text{(identity)}$$

They are of completely different character.

Assign random link variables on a periodic cubic lattice and sum the six plaquettes bounding an elementary cube with orientation.

Field type	Max residual
non-compact (random, $L = 6$, seed 0)	3.55×10^{-15}
seed 1	2.67×10^{-15}
seed 2	2.22×10^{-15}

Exactly zero (rounding only). A combinatorial fact independent of dynamics.

$\nabla \cdot \mathbf{B} = 0$ is not a law to impose but an identity that follows, even on a lattice.

21-1. But compactification lets monopoles in 🆕

Reduce each plaquette to the fundamental branch before summing over the cube, and the residual is **always a multiple of N**.

Group	Values of reduced (dF)	Fraction of cubes carrying a monopole
$\mathbb{Z}_2$	$\{-2, 0, +2\}$	0.361
$\mathbb{Z}_3$	$\{-3, 0, +3, +6\}$	0.528
$\mathbb{Z}_5$	$\{-10, -5, 0, +5, +10\}$	0.500

Magnetic monopoles exist naturally as CA defects. Forbidding them needs an extra condition.

Monopoles are not observed in our universe. **In a CA universe monopoles are something to remove, not to make** — the sign is reversed relative to reality.

Chapter 22 Charge becomes $\mathrm{mod}\ N$ 🆕

— Why does charge come only in integer multiples

The $\mathbb{Z}_N$ Gauss law is $q_j \equiv E_j - E_{j-1} \pmod N$. Enumerating every configuration on a ring of L links (a closed universe):

N	L	total charge $\mathrm{mod}\ N$	as an integer
2	4	$\{0\}$	$\{0, 2, 4\}$
3	4	$\{0\}$	$\{0, 3, 6\}$
5	3	$\{0\}$	$\{0, 5, 10\}$

"How many electrons are there in the universe" is not a question a $\mathbb{Z}_N$ theory possesses.

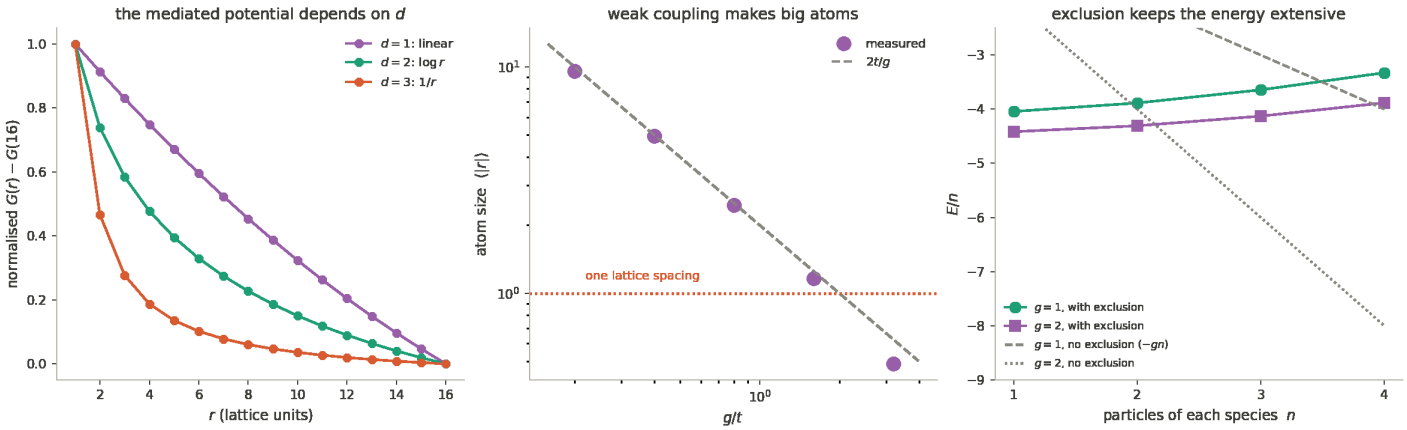


Figure 22-1. Structure of $\mathbb{Z}_N$ charge; bilinears cannot carry charge ≥ 3.

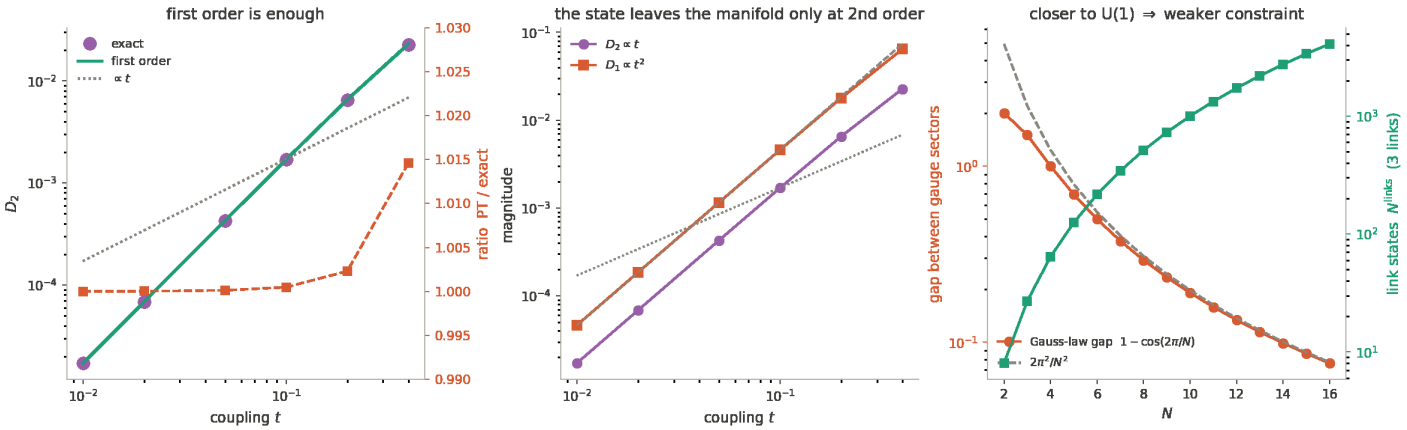


Figure 22-2. Perturbative expansion in $\mathbb{Z}_N$.

Chapter 23 $U(1)$ cannot be put in ❌ / but emergence is another matter ⚠️

— What Maxwell's equations describe, and where the CA fails to carry them

One would like to take $N \to \infty$. **It degrades twice.**

Degradation	Content
weakening constraint	the sector gap shrinks as $2\pi^2/N^2$
exploding resources	the number of states is N^{links}

There is also an algebraic obstruction ✅ — **Majorana bilinears cannot carry $\mathbb{Z}_N$ charge for $N \geq 3$.**

23-1. But "cannot emerge" does not follow ⚠️📚

An exact microscopic $U(1)$ cannot be written into a locally finite-dimensional CA ❯

But a $U(1)$ gauge field **can emerge** at low energy 🤔

📚 In 3+1 dimensional quantum spin ice ($S = 1/2$ on the pyrochlore lattice), with local Hilbert spaces of dimension 2, an **emergent $U(1)$ gauge field and "photon" appear as low-energy excitations** (Hermele–Fisher–Balents 2004 and others).

What a CA cannot have is $U(1)$ as a **fundamental** law.
An **emergent** $U(1)$ is not forbidden — but needs $d \geq 3$ and suitable constraints.

An important relaxation for the wish. That electromagnetism is absent from the fundamental law is not fatal for a discrete foundation; on the contrary, an emergent electromagnetism is consistent with many modern frameworks.

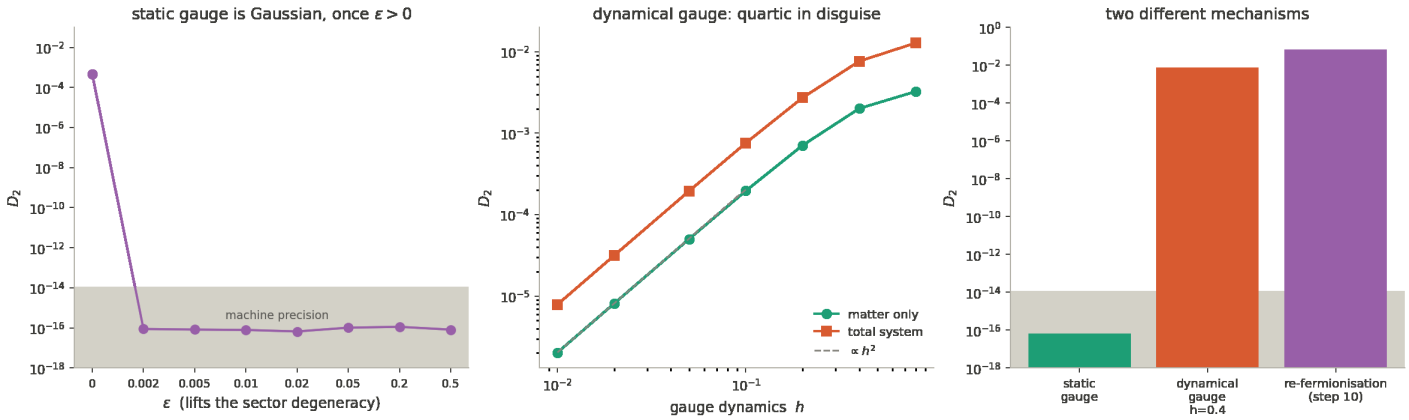


Figure 23-1. Structure of a gauge field on the lattice.

Part VII Spacetime — relativity

Chapter 24 Lorentz invariance — two completely different violations 🆕

— How far does the constancy of light survive on a lattice

The greatest obstacle for a CA, and **frequently misunderstood**. There are two kinds and their severity differs enormously.

24-1. Dimension-6 violation: dispersion — harmless 🆕

$$\omega^2 = \sum_i \frac{4}{a^2}\sin^2\frac{k_i a}{2}$$

At fixed $|k|$ the speed depends on direction.

ka	axis	diagonal	anisotropy
1.0	-4.115×10^{-2}	-1.383×10^{-2}	-2.732×10^{-2}
10^{-1}	-4.166×10^{-4}	-1.389×10^{-4}	-2.777×10^{-4}
10^{-3}	-4.167×10^{-8}	-1.389×10^{-8}	-2.778×10^{-8}

$$\text{anisotropy} = \frac{(ka)^2}{36} \qquad (\text{measured } 2.777778\times10^{-2},\ \text{analytic } 1/36)$$

At a Planck-length lattice this is entirely harmless.

Photon	ka at $a=\ell_P$	anisotropy
1 eV	8.2×10^{-29}	$\sim10^{-58}$
100 TeV	8.2×10^{-15}	$\sim10^{-30}$

The naive criticism "a lattice breaks Lorentz invariance" misses, as far as this term goes.

> ⚠️ **Note (thirteenth edition).** The "anisotropy" of the table is axis minus body diagonal and negative; the formula is its absolute value. §87-9.

24-2. Dimension-4 violation: species-dependent limiting speeds — the real problem ❌🆕

Give two fields different couplings and $v_i \propto t_i$; the ratio is independent of the lattice spacing. And it **does not flow at all** under coarse-graining.

Block size	1	2	8	64
$\|v_1/v_2 - 1\|$	9.900990×10^{-3}	same	same	same

A mismatch of limiting speeds is a dimension-4 operator and is unsuppressed in the infrared. Lorentz invariance requires **tuning each species by hand**.

Experimental bounds are $\delta c/c \lesssim 10^{-20}$ (for some species 10^{-22}). 📚 This is not specific to lattices but the general **naturalness problem** of theories without Lorentz symmetry.

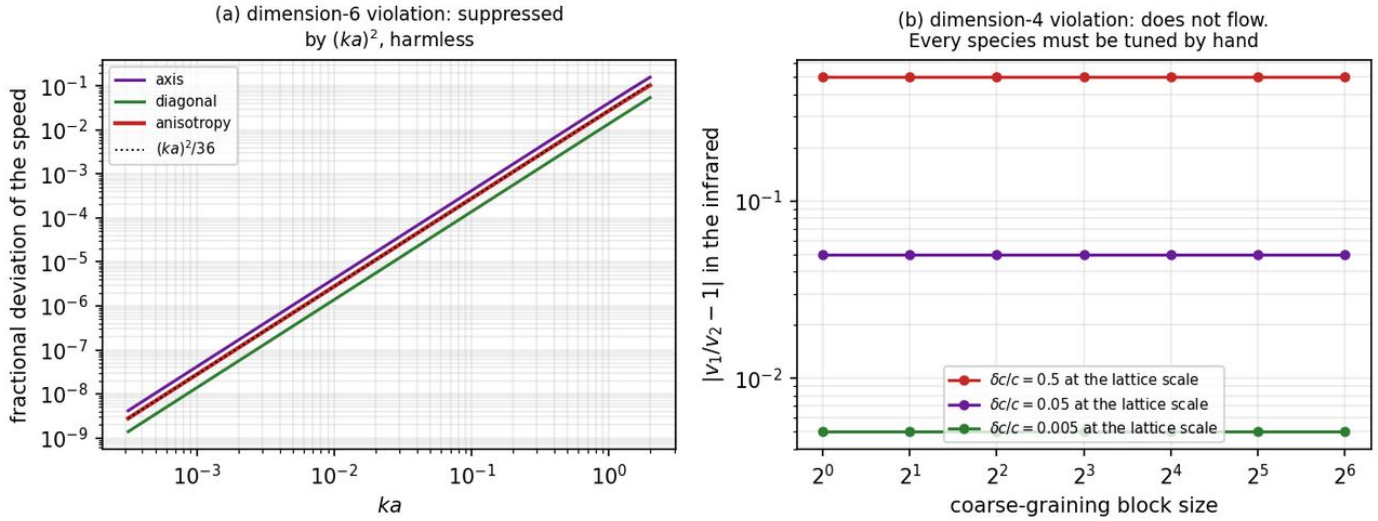


Figure 24-1. 🆕 (a) Dimension-6 violation is suppressed as $(ka)^2/36$ and harmless on a Planck lattice. (b) Dimension-4 violation does not flow at all under coarse-graining. That is the real obstacle.

24-3. There is only one way out 💭

> A regular lattice is incompatible in principle with Lorentz invariance.
> To keep discreteness and be Lorentz invariant, the discreteness must be **irregular**.

The reason is simple: **a regular lattice picks out a preferred inertial frame** — the one in which it is at rest.

By contrast, **Poisson sprinkling** points into Minkowski space gives statistics invariant under Lorentz transformations. 📚 That is the starting point of causal set theory.

> Randomness buys Lorentz invariance.

For a reader who wants a simple law without determinism, this is a decisive coincidence. Randomness is not an extra; it is **required** for a discrete world to be Lorentz invariant. Part XI follows this thread.

> ⚠️ **Restriction (thirteenth edition).** This section is about **exact** Lorentz invariance. Effective low-energy invariance on a regular lattice (critical points, symmetry protection, fine tuning) is possible in principle. "Necessary" is too strong. §87-9.

Part VIII Matter, dimension, and the mass ratio

Part II found that mass is geometry, but **the ratio of masses remained an irreducible input**. This part digs into it. **It is the longest part of the book and contains the most failures.**

Chapter 25 $d = 3$ is demanded three times, independently ✅

— Why is space three-dimensional

25-1. Atoms need $1/r$

Measured lattice Green functions:

Dimension	$V(r)$	R^2
$d = 1$	**linear** (confining)	0.997
$d = 2$	$\log r$	0.9996
$d = 3$	$\mathbf{1/r}$	0.999

Bound-state size $R^*/a \sim 2t/g$ (measured ratio 0.96–0.99).

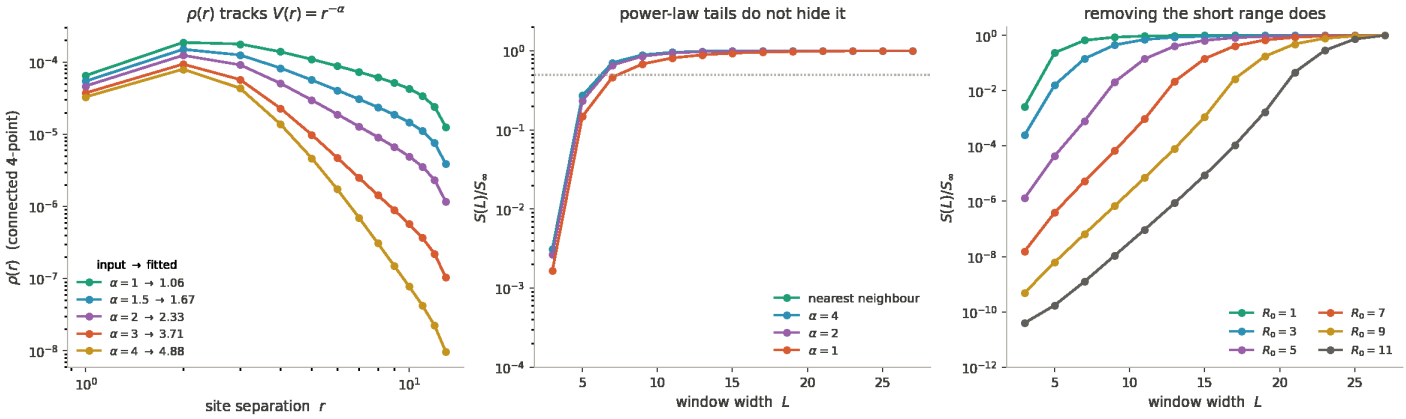


Figure 25-1. Long-distance behaviour by dimension.

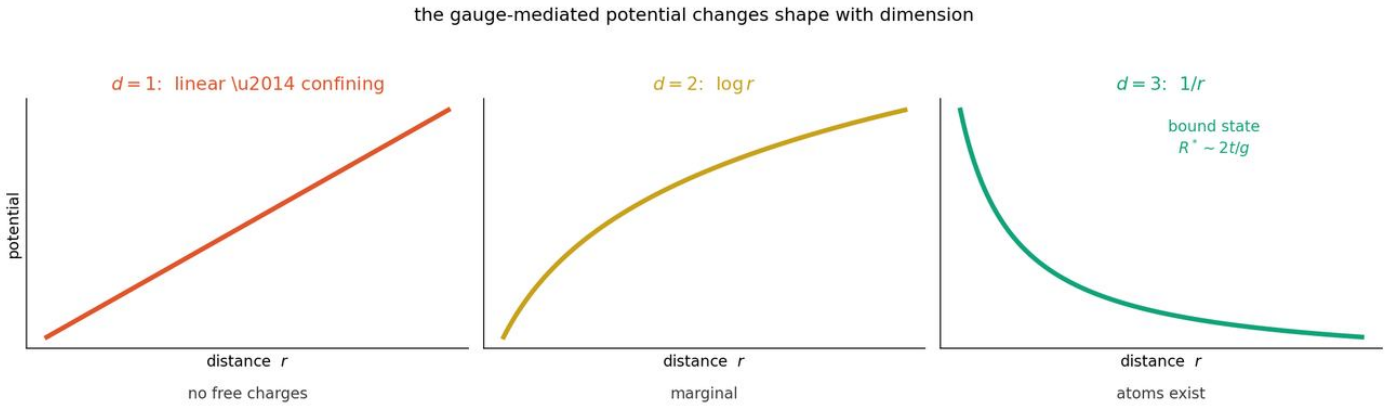


Figure 25-2. What the dimension decides.

25-2. Three independent routes arrive at the same place

Requirement	Dimension	Reason
atoms	$d = 3$	$1/r$ potential
deconfined phase	$d \geq 2$	finite Mott gap
Cooper pairs	Fermi surface	relevant channel

$$\boxed{\text{For atoms to exist, space must be three-dimensional. The same holds in a CA.}}$$

This is not a defect of CA but agreement with reality.

Chapter 26 The mass-ratio question — three modes of explanation

— Why does the electron-to-proton mass ratio have the value it has

$$\boxed{\begin{aligned} \textbf{Filtering}&: \text{ the ratio is free, but only that value permits atoms} \\ \textbf{Generation}&: \text{ the ratio is not free; some mechanism } \textbf{made} \text{ it} \\ \textbf{Selection}&: \text{ all ratios existed initially and the unsustainable ones died out} \end{aligned}}$$

One must not decide in advance which is needed. Eliminate in order of cheapness, and **introduce the next only for the residue**.

$$\text{filtering (measurable at once)} \;\to\; \text{generation (does an existing mechanism suffice?)} \;\to\; \text{selection (last)}$$

> 💭 In the reverse order one would push onto selection what generation could have explained.

Chapter 27 Filtering — only a lower bound emerges 🆕

— How much can anthropic reasoning actually fix a number

27-1. Why the naive guess fails

$$\mu = m_1 \frac{r}{1+r}$$

$m_1/2$ at $r = 1$, m_1 as $r \to \infty$. **The reduced mass moves by a factor of two in total.** Even if the ratio ranges from 1 to 10^4, the atom's size changes by two.

27-2. Looking for a lattice-specific route 🆕

On a lattice, centre of mass and relative coordinate **do not decouple**. With $k_1 = P/2 + q$, $k_2 = P/2 - q$,

$$K(P,q) = \sum_d \left[-2A_d \cos q_d + 2B_d \sin q_d\right], \qquad \boxed{B_d = (t_1 - t_2) \sin \frac{P_d}{2}}$$

The B term vanishes at $t_1 = t_2$ and at $P = 0$, and nowhere else. **It is the lattice's memory of mass asymmetry.**

Hypothesis: at extreme ratios, a moving atom is torn apart by the lattice.

27-3. Calibration

Test	Result
$\|H - H^\dagger\|$ at $t_1 = t_2$	0.00 (B vanishes identically)
binding energy at $g = 0$	-1.7×10^{-1} (box artefact; **distinguishable from real binding**)

27-4. Result — everything saturates by $r \sim 32$ ❌🆕

r	$\mu/\mu(1)$	E_b	R^*/a	$M^*/(m_1+m_2)$
1	1.0000	1.4788	2.2432	1.0917
8	1.7778	2.0515	1.5773	1.1856
32	1.9394	2.1383	1.5059	1.2057
512	1.9961	2.1668	1.4835	1.2128
2048	1.9990	2.1683	1.4824	**1.2131**

The composite's inertia **saturates** at 1.213 — had it become trapped it would have diverged. **The hypothesis is refuted.**

Furthermore the binding **strengthens** with momentum ($r = 1$: $1.48 \to 2.34$ at $P = \pi$). **A moving atom does not dissociate.** Recorded as a null result.

27-5. The only surviving constraint is one-sided ⚠️🆕

$$\frac{W_{\text{heavy}}}{E_b} = \frac{6}{r\,E_b} \quad \text{(validity of Born–Oppenheimer)}$$

r	1	4	8	16	32
ratio	4.06	0.77	0.37	**0.18**	0.09
BO valid	❌	❌	❌	✅	✅

$r \gtrsim 10$ is required. There is no upper bound.

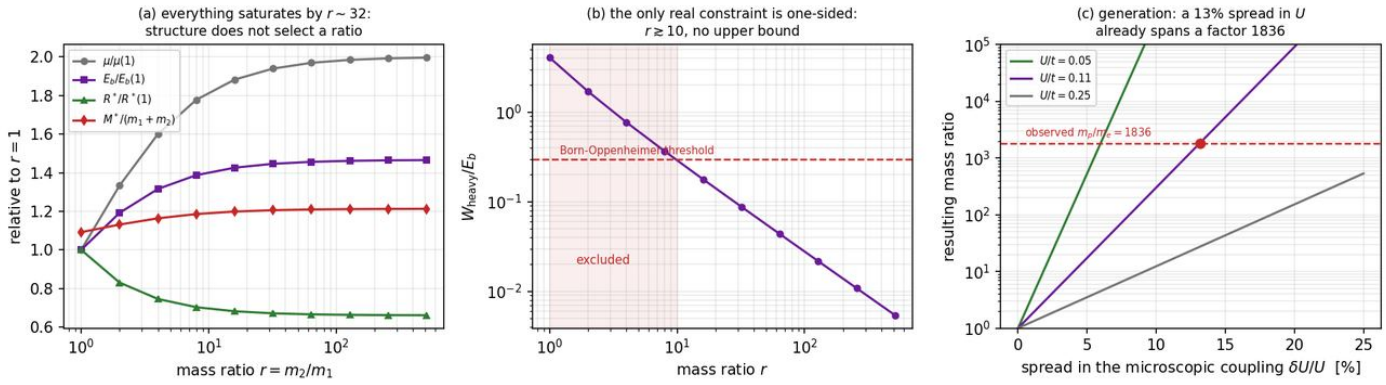


Figure 27-1. 🆕 (a) Reduced mass, binding energy, radius and inertia all saturate by $r \sim 32$. (b) The only constraint is a one-sided lower bound. (c) Exponential amplification by dimensional transmutation (Chapter 28).

Filtering yields only the one-sided bound $r \gtrsim 10$.
"Only that ratio can persist" is not supported by atomic structure.

This negative result is the most valuable step of the part. Before introducing selection, it fixed what selection must explain.

Chapter 28 Generation — the value can be made 🆕

— How can a rule manufacture a huge ratio

Use Chapter 13's dimensional transmutation. The sensitivity is $d\ln\Delta_c/d\ln U = 2\pi t/U$.

U/t	$2\pi t/U$	$\delta U/U$ **needed for** 1836
0.05	125.66	5.98%
0.11	**57.12**	**13.16%**
0.25	25.13	29.90%

A 13% difference in a microscopic coupling produces 1836 outright.

Explaining the magnitude needs neither selection nor anthropics. But something remains unexplained.

Why this ratio? ⟶ Why is U the same everywhere in the universe?

The ratio problem has been reduced to the **homogeneity** problem.

Only now is there a reason to introduce selection.

Chapter 29 Selection — does the wall move? 🆕

— Can the constants of nature drift with time

29-1. Two failures (recorded)

Version 1: the initial dimerisation was not the true Δ_0, and the whole run was dominated by breathing. Calibration failed; discarded.

Version 2: relaxing the kink while re-diagonalising the fermions at every step hit the wall's **zero mode** (Jackiw–Rebbi); the Schur branch jumped and the kink dissolved. **The same failure mode recorded six times in Appendix C.** Diagnosis kept, code discarded.

Version 3 (adopted): reduce the fermions exactly to $V(\Delta)$. No zero-mode ambiguity, and energy conservation checkable to machine precision.

29-2. Calibration (seven items, all passed)

Test	Result
Hellmann–Feynman	1.4×10^{-9}
degeneracy of the two vacua ($h=0$)	5.6×10^{-17}
static kink residual	1.4×10^{-15}
wall does not move at $h=0$	3.4×10^{-13} bonds
energy conservation at $h=0$	2.1×10^{-16}

29-3. Result 1 — the wall moves even in a closed reversible system ❌🆕

t	closed	expanding (fixed cells)	expanding (cells added)	damped
0	199.5	199.5	199.5	199.5

t	closed	expanding (fixed cells)	expanding (cells added)	damped
750	111.7	187.4	112.6	178.6
1500	3.2	187.4	4.1	156.0
6000	~ 10	187.5	~ 8	~ 20

The reasoning "the released energy has nowhere to go" was wrong. It goes into the wall's own kinetic energy, and then into the radiation modes of the field. Extended to $t = 12000$ the wall does not return (energy conserved to 4.1×10^{-7}).

> Selection needs no external bath. **The system's own many modes are the bath.**
> Same structure as Chapter 16 — the arrow is in the **partition**, and here the partition arises spontaneously.

By-product: the worry that "making the rule stochastic for selection would break Bell" is void. There was never a need to make it stochastic.

29-4. Result 2 — expansion pins the wall ⚠️🆕

Expanding at fixed cell count shrinks the comoving kink width as $1/a$ below the lattice spacing, and **Peierls–Nabarro pinning** sets in (stops after 12 bonds).

> In a CA with a fixed number of cells, expansion pushes structure below the lattice scale and **pins its own defects. A CA cannot add cells.**

29-5. Result 3 — selection requires a minimum drive ❌🆕

J	kink width	h_c	drive per bond
0.4	3.836	4.157×10^{-3}	3.453×10^{-3}
1.0	6.246	1.868×10^{-3}	1.552×10^{-3}
6.0	15.595	4.809×10^{-4}	3.995×10^{-4}

$$h_c \propto w^{-1.55} \quad (w = 3.8\text{–}15.6)$$

Confirmed static: varying the observation time by a factor 8 leaves h_c unchanged to all digits.

> Below h_c the wall **never** moves, however long one waits.

With friction the terminal velocity is $v = 0.30\,h/\nu$ (within 15%).

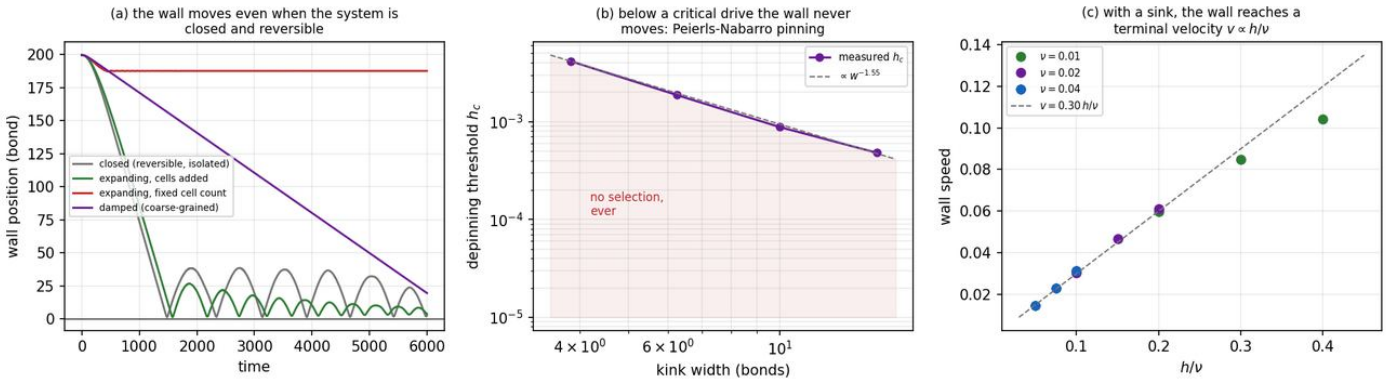


Figure 29-1. 🆕 (a) One initial condition, four settings. Even the closed reversible system (grey) sweeps. Only fixed-cell expansion (red) stops. (b) The depinning threshold; below the red region selection never happens. (c) Terminal velocity when a sink exists.

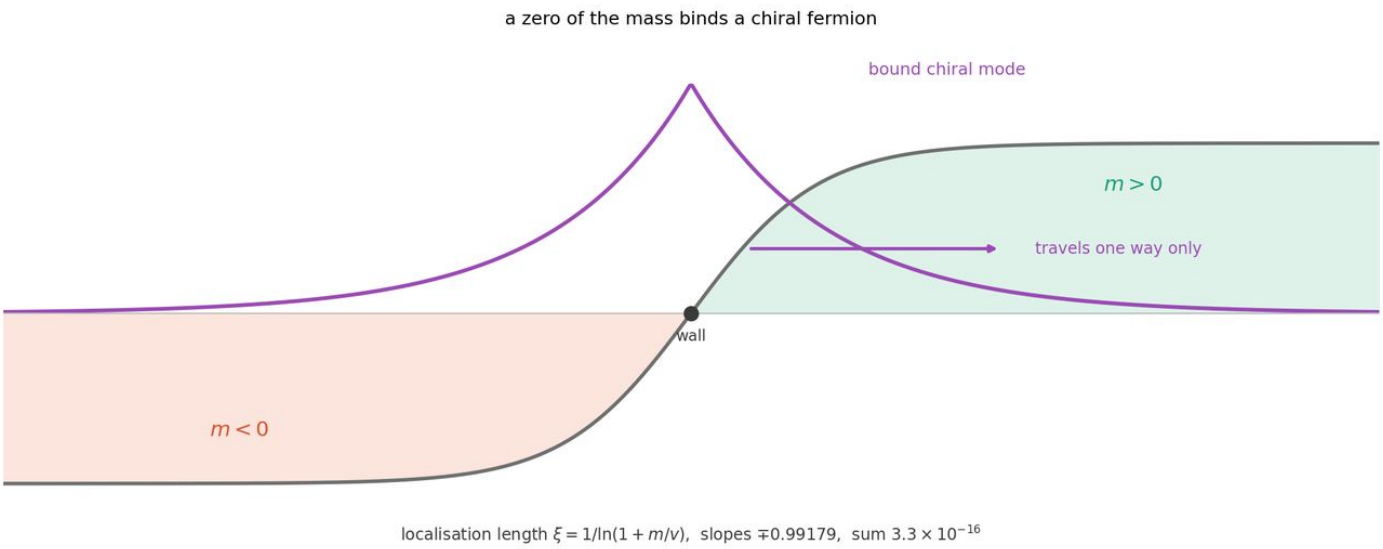


Figure 29-2. Structure of a domain wall.

29-6. The resolution of selection 🆕

Combining the threshold with the drive from domain competition ($d\varepsilon/d(m/v) \to -0.25$ per bond):

$$0.25\,\delta > 1.2 \times 10^{-3} \quad \Longrightarrow \quad \boxed{\delta(m/v) \gtrsim 5 \times 10^{-3}}$$

Regions whose ratios differ by less than 0.5% **coexist forever**. Selection has a resolution limit.

Chapter 30 The horizon — a gate and a clock 🆕

— Does the cosmic horizon fix physical quantities

As expansion proceeds a horizon appears locally and distant interactions stop arriving. Does this affect the mass ratio? **It does — and in two opposite ways.**

Face	Quantity	Effect
infrared cutoff	$L_H = c/H$	order with $\xi > L_H$ cannot grow
horizon temperature	$T_{\rm dS} = H/2\pi$	thermal fluctuations melt order

30-1. The infrared cutoff compresses the ratio 🆕

N	$\Delta_0(K{=}0.45)$	$\Delta_0(K{=}1.00)$	ratio	ratio/ratio(∞)	N/ξ
12	0.5100	0.1533	3.326	**0.562**	1.69
32	0.4500	0.1065	4.225	0.713	4.50
128	0.4221	0.0776	5.441	0.919	18.01
300	0.4166	0.0703	5.923	1.000	42.20

Dimensional transmutation only becomes small by flowing to the infrared, so **stopping the flow leaves the value large**. And the longer the correlation length, the stronger the lift.

A horizon compresses mass ratios towards 1.

For $N/\xi \gtrsim 20$ the ratio is within 4%.

30-2. The horizon temperature makes the ratio diverge 🆕

T	$\Delta_0(0.45)$	$\Delta_0(1.00)$	ratio
0.000	0.4227	0.0784	5.394
0.080	0.4227	0.0602	7.020
0.120	0.4222	0.0361	11.698
0.250	0.3944	0.0128	30.736

The smaller scale melts first. Heat destroys the weakest hierarchy first.

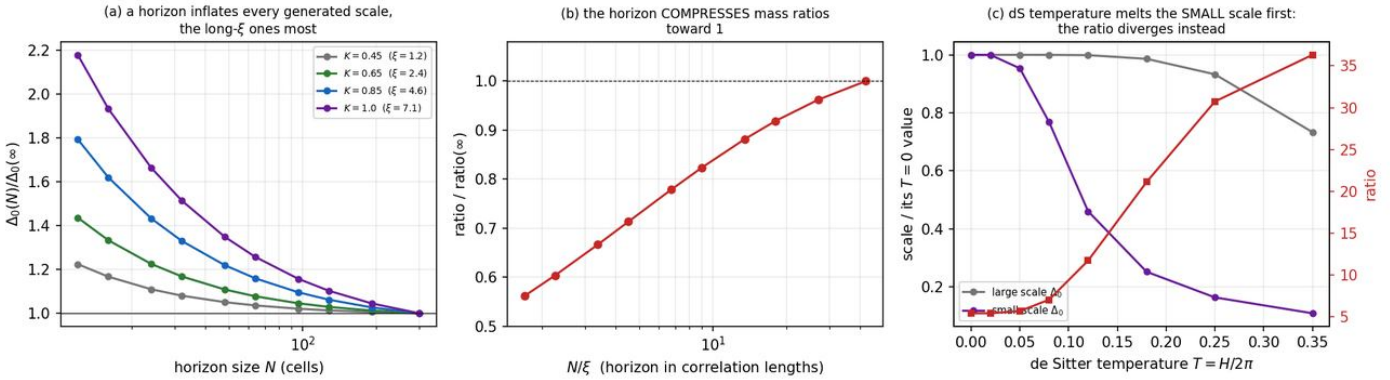


Figure 30-1. 🆕 (a) A horizon lifts every generated scale, the long-ξ ones most. (b) Hence the ratio is compressed towards 1. (c) De Sitter temperature instead melts the small scale first and makes the ratio diverge.

30-3. Unified, both say the same thing

$$H \lesssim \Delta \qquad \text{(the Hubble rate must be below the scale being generated)}$$

(1) $H \gg \Delta_{\text{large}}$: no order; no ratio exists
(2) $H \sim \Delta_{\text{large}}$: the large scale freezes
(3) $H \sim \Delta_{\text{small}}$: the small scale freezes; **the ratio is now defined**
(4) domain competition inside the horizon (Chapter 29)
(5) regions beyond the horizon **never communicate**

The horizon is not a selector. **It is a gate and a clock.**

Chapter 31 Is patch uniqueness enough?

— Does a local rule determine the whole universe

Patch uniqueness : uniqueness inside our horizon suffices; other ratios may lie outside
Global uniqueness : the ratio is the same everywhere; a consistency condition fixes it

31-1. What patch uniqueness requires 🆕

The wall's **surface tension** turned out to have an invariant.

$$\frac{\sigma}{|V(\Delta_0)| \cdot w} = 0.037 \pm 0.001 \qquad \text{(unchanged over a factor 4 in width)}$$

A wall is "a membrane of thickness 3.7% of the bulk". $\sigma = 0.0932$ at $J = 1.5$.

The coexistence dilemma — combining with Chapter 29-5:

$$\boxed{\begin{array}{l}\delta(m/v) < 0.48\% \ : \text{ the wall is pinned and coexistence is permanent} \\ \delta(m/v) > 0.48\% \ : \text{ the wall depins and } \textbf{one domain eats the other}\end{array}}$$

$$\boxed{\begin{array}{l}\text{Domains that coexist stably have nearly the same ratio.} \\ \text{Domains whose ratios genuinely differ do not coexist stably.} \\ \text{Hence "other ratios exist" holds } \textbf{only beyond the horizon}.\end{array}}$$

The horizon is not a passive excuse but an active precondition for other ratios to exist at all. And it protects permanently only if the comoving horizon shrinks, i.e. only if $\Lambda > 0$.

31-2. What global uniqueness requires — rigidity

$$\text{rigidity} \iff \frac{d \ln \Delta_0}{dK} \simeq 0 \text{ on a finite interval}$$

The unconstrained model has no rigidity.

K	0.45	0.65	0.85	1.05
$d \ln \Delta_0 / dK$	-3.916	-3.354	-3.064	-2.802

Not one interval approaches zero; a 1% change in K moves the ratio by 0.6%.

$$\boxed{\text{In this model the ratio is a continuum. No value is preferred.}}$$

What is refuted is the story that dimensional transmutation alone yields uniqueness. Only one candidate ingredient remains — **constraints**.

Chapter 32 The search for rigidity — two kinds, measured separately 🆕

— How rigid is the theory — how much can be moved

$$\boxed{\begin{array}{l}\textbf{Type I}: \text{ the generated scale is independent of a continuous coupling (a plateau)} \\ \textbf{Type II}: \text{ the coupling itself takes only discrete values (a ladder)}\end{array}}$$

Asymmetry declared in advance: a jump is decisive; no jump says nothing (the search is semi-decidable).

32-1. Type I, first search — sector classification ❌🆕

Calibration failed once (the "local charge" was not a Gauss law: $|[H, G_j]| = 4.0$). Rebuilt as a genuine $\mathbb{Z}_2$ lattice gauge theory and passed seven checks (physical sector dimension 64; n_A conserved at $v = 0$, broken at $v \neq 0$ with the total surviving).

Scanning r at 25 points found **exactly one special ratio**.

$$r = 1 \quad \text{with} \quad \dim : 288 \to 295 \quad (+7)$$

Same at tolerance 10^{-9} and 10^{-6}; no jump at $r = 0.999$ or 1.001; unchanged when g is varied by 4 and v by 8. The unconstrained control shows no jump.

$$\boxed{r = 1 \text{ is where the two species are identical — the point at which } \textbf{no mass ratio exists}.}$$

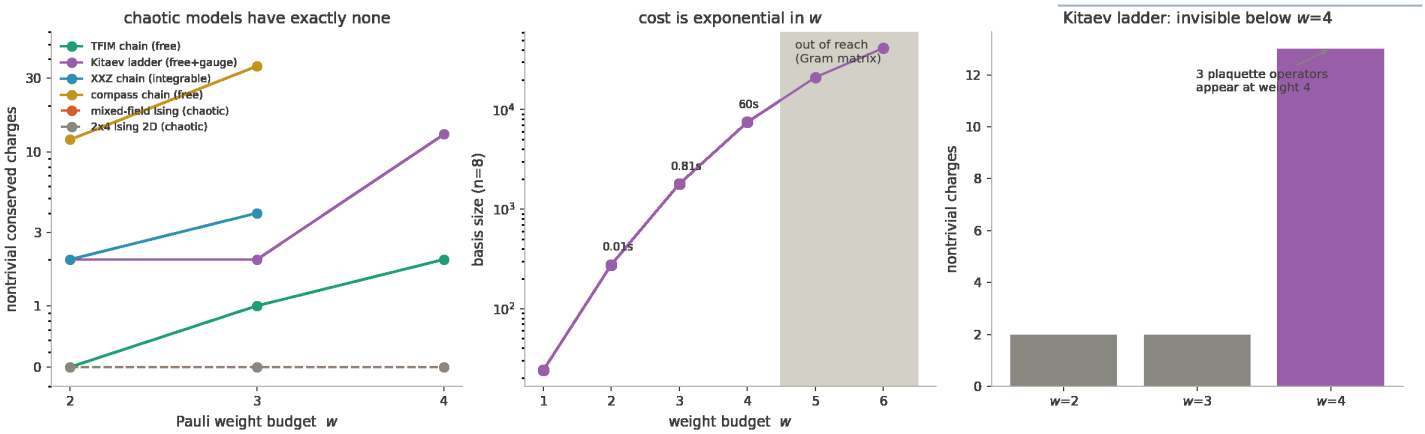


Figure 32-1. Exhaustive search for conserved quantities. Weight $w \leq 4$ is the practical limit.



Figure 32-2. Reading conserved quantities off the kernel of the commutator map; the monomial/polynomial classification is decided here.

32-2. Type I, second search — $d \ln \Delta_0 / dK$ under a constraint ❌🆕

Calibration failed twice (the minimum sat at the edge of the search range, so the "plateau" was a truncation artefact). Fixed by widening the range.

Model	smallest non-zero Δ_0 realised
ungauged (control)	0.0575
gauged $h = 1.0$	0.0963
gauged $h = 1.5$	0.1525

No qualitative difference. There is no Type I rigidity.

But exact flatness does appear — at zero. For $h \geq 1.8$, $E(\Delta)$ is exactly Δ-independent over a finite range (residual 10^{-15}, $E(0) = -hL$ exactly), because the Gauss-sector vacuum contains no matter and the gauge-invariant hopping annihilates it. The threshold is $\Delta_c \simeq 0.79h - 0.94$.

32-3. The same shape, three times 💭

§ 32-1: the only ratio singled out is $r = 1$ — where no ratio exists
§ 32-2: exact rigidity occurs only at $\Delta_0 = 0$ — where no scale exists
Chapter 12: the number of bilinears commuting with a constraint is **zero**

A constraint states exactly what is impossible. It does not state what is realised.

32-4. Type II — the coupling itself is quantised ✅🆕

From the $\mathbb{Z}_N$ electric term and the Gauss law, the cost per link between a charge pair q is

$$\boxed{U_q(N) = 2h\big[1 - \cos(2\pi q/N)\big]}$$

Measured directly from explicit flux configurations without using the formula.

N	q	measured	formula	difference
5	1	0.9673762079	0.9673762079	3.3×10^{-16}
5	2	2.5326237921	2.5326237921	4.4×10^{-16}
7	3	2.6613564151	2.6613564151	4.4×10^{-16}

h and t are continuous but $u_q = U_q/2h$ **is fixed by the integers** (N, q) **alone**.

N	q=1	q=2	q=3	q=4
5	0.6910	1.8090	—	—
7	0.3765	1.2225	1.9010	—
9	0.2340	0.8264	1.5000	1.9397
12	0.1340	0.5000	1.0000	1.5000

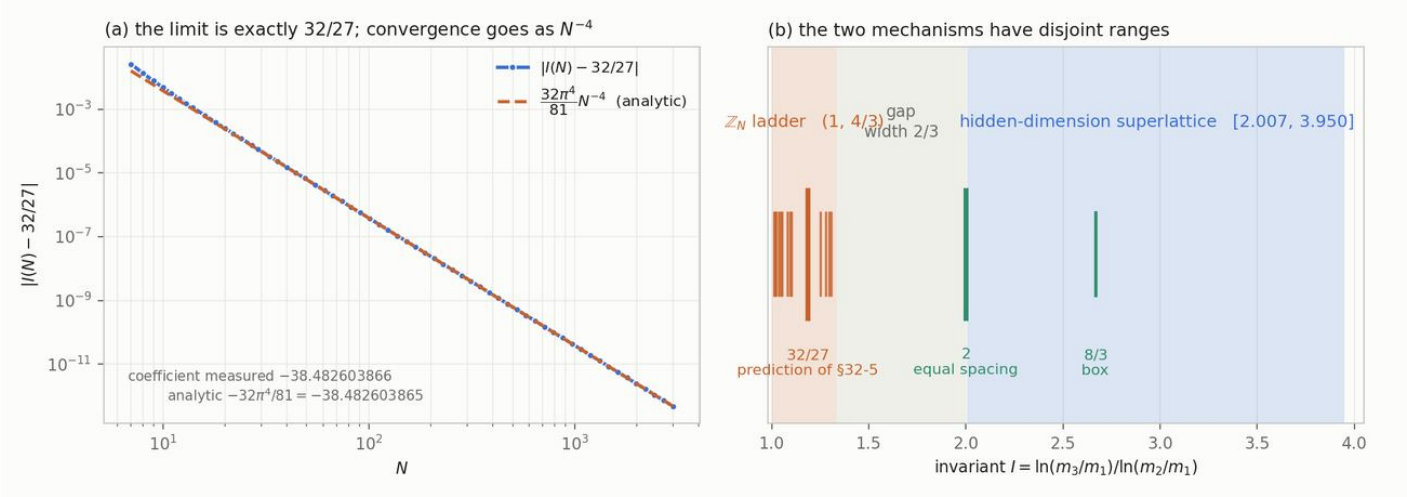
32-5. With three species the sliding cancels ✅🆕

With $\lambda = \pi c t/h$ ($c = 0.968$), $\ln \Delta_q = -\lambda/u_q$. The whole ladder slides with λ, but with three species:

$$\boxed{\frac{\ln(m_3/m_1)}{\ln(m_2/m_1)} = \frac{R(N;1,3)}{R(N;1,2)} \quad \text{— a pure number, independent of } t, h \text{ and overall scale}}$$

N	7	9	12	15	20	$\to \infty$
invariant	1.15883	1.17736	1.18301	1.18435	1.18493	**1.18517**

> **Correction (Edition 3.0).** The limit is **not** 1.18517. Re-running the ladder at higher precision gives a closed form. With $A = 4\cos^2(\pi/N)$,
>
> $$I(N) = \frac{A^2(A-2)}{(A-1)^3}, \qquad I(\infty) = \frac{32}{27} = 1.185185\ldots$$
>
> and the approach obeys $|I(N) - 32/27| = \frac{32\pi^4}{81} N^{-4}$ — measured coefficient -38.482603866 against the analytic $-32\pi^4/81 = -38.482603865$. The number 1.18517 tabulated above is $I(N \simeq 40)$, not the limit.
>
> My declared convergence exponent was -2; the true exponent is -4, because the q-independent $\frac{1}{3}$ in $\csc^2 x = x^{-2} + \frac{1}{3} + \frac{x^2}{15}$ cancels in the ratio R. The prediction was wrong, not the model.
>
> The same invariant also **separates the two mechanisms**, which the density argument below could not: the $\mathbb{Z}_N$ ladder family lives in $(1,\ 4/3)$ and the hidden-dimension superlattice in $[2.007,\ 3.950]$ — disjoint, with a gap of width $2/3$. **→ Chapter 33.**



The **first non-trivial prediction** obtained on the side of global uniqueness.

Against observation: the t/h needed for 1836 is $O(1)$ across the board (best: $N = 15$, $q = 2, 6$, giving $\mathbf{1.0007}$). **No fine tuning.**

But the ladder is too dense ⚠️ — 112 rungs for $N \leq 15$, median spacing 0.109 in $\ln$, four rungs within a factor 1.5 of 1836. **Discrete, but not sparse enough to predict.**

Chapter 33 A hidden dimension — tetrahedra, diamond, superlattice ⚠️🆕

— Can the mass hierarchy be built by a second mechanism

The ladder invariant of §32-5 claimed it could **separate two mechanisms**. This chapter is the second one. Where Chapter 32 built a mass ratio from a $\mathbb{Z}_N$ ladder, here it is built from **an extra, hidden direction in space**.

The conclusion first.

(i) The anisotropy of a tetrahedral stencil starts at k^3; two sublattices (diamond) push it to k^4.
(ii) The low-energy excitations of nearest-neighbour diamond are **nodal lines**, not Dirac points.
(iii) Read as a superlattice, a hidden dimension produces three masses from one mechanism.
(iv) Its invariant is confined to $I \geq 2$ and **cannot reach** the $32/27$ of § 32-5.

The two mechanisms predict disjoint ranges for the same dimensionless number. They are incompatible. One or the other.

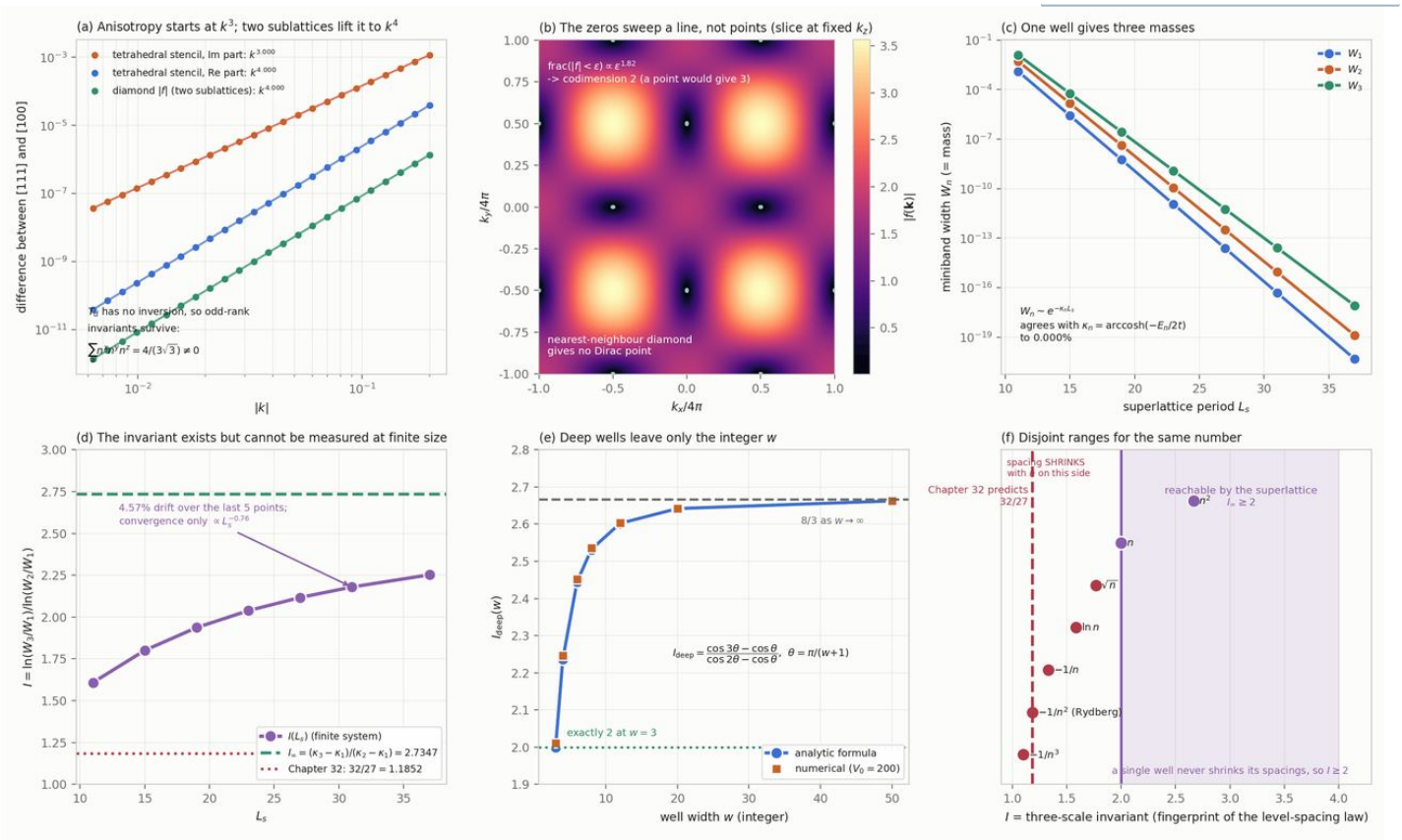


Fig. 33-1. (a) Order of the anisotropy. (b) The zero set is a line. (c)(d) Superlattice minibands and the invariant. (e) Quantization in deep wells. (f) The ranges of the two mechanisms.

33-1. Tetrahedra are isotropic to second order and fail at third

The four tetrahedral vectors $\mathbf{n}_a$ satisfy exactly

$$\sum_a \mathbf{n}_a = 0, \qquad \sum_a n_a^i n_a^j = \tfrac{4}{3}\delta^{ij}$$

(residuals 0 and 4.4×10^{-16}). So far the usual argument for "isotropy at second order" is correct. **But the third-rank tensor had never been tested.**

$$\sum_a n_a^x n_a^y n_a^z = \frac{4}{3\sqrt{3}} = 0.769800358919501\ (\neq 0), \qquad \sum_a n_a^x n_a^x n_a^y = 0$$

The tetrahedral point group T_d **contains no inversion**, so odd-rank invariants survive. The cubic term of the long-wavelength expansion is $\frac{i}{6} \cdot \frac{8}{\sqrt{3}} k_x k_y k_z$: maximal along $[111]$ and exactly zero along $[100]$. Measured exponents (Fig. (a)):

object	exponent of $[111] - [100]$
tetrahedral stencil (one sublattice), Im part	$k^{3.000}$
tetrahedral stencil, Re part	$k^{4.000}$
diamond $\lvert f\rvert$ (two sublattices)	$k^{4.000}$

Anisotropy starts at k^3; the A, B sublattices restore inversion and lift the spectrum to k^4.

This gives a quantitative answer to "why diamond rather than a naive four-neighbour tetrahedral update" — the bipartite structure buys one order of anisotropy.

> 💭 Note that k^3 belongs to **neither class** of Chapter 24 (dimension 6 is harmless at $(ka)^2/36$; dimension 4 does not run). Whether it runs under coarse-graining is untested.

33-2. Nearest-neighbour diamond has no Dirac point ❌

$$f(\mathbf{k}) = \sum_a e^{i\mathbf{k}\cdot\mathbf{d}_a} = 4\,[\cos u \cos v \cos w - i\,\sin u \sin v \sin w], \qquad u = k_x/4, \dots$$

(residual of the closed form 4.4×10^{-16}). Vanishing requires $\cos u \cos v \cos w = 0$ **and** $\sin u \sin v \sin w = 0$ — **two** conditions, which in three dimensions sweep out a **line**. Numerically,

$$\mathrm{frac}(|f| < \varepsilon) \propto \varepsilon^{1.82} \qquad (4 \times 10^6 \text{ points})$$

giving codimension $\simeq 2$ (a point would give 3).

> Nearest-neighbour diamond does not give Weyl or Dirac particles. It gives nodal lines.

"Use the sublattice degree of freedom as a spinor index" is not enough; this is exactly why Fu–Kane–Mele needed spin–orbit coupling. Moreover **the group velocity along a nodal line vanishes**, so the low-energy excitations are not isotropic to begin with — the k^4 of §33-1 is about the Γ point (the band edge), not about the low-energy region. **The claim "a tetrahedral lattice is isotropic at low energy" is not testable until one says around which point.**

33-3. The hidden dimension as a superlattice — three masses from one well

Place a periodic array of wells of period L_s, width w and depth V_0 along the hidden direction. Tunnelling between walls makes minibands, and their widths become visible masses.

$$m_n \equiv W_n = 4|t_{\mathrm{eff}}^{(n)}| \sim e^{-\kappa_n L_s}$$

Measured in 60-digit arithmetic (Fig. (c)):

L_s	W_1	W_2	W_3
11	1.28120×10^{-3}	5.28781×10^{-3}	1.25338×10^{-2}
19	5.58211×10^{-9}	4.12583×10^{-8}	2.68697×10^{-7}
27	2.43211×10^{-14}	3.21915×10^{-13}	5.76042×10^{-12}
37	4.84133×10^{-21}	1.32748×10^{-19}	8.40315×10^{-18}

$$\kappa_1 = 1.542967, \quad \kappa_2 = 1.470134, \quad \kappa_3 = 1.343792$$

each agreeing with $\kappa_n = \operatorname{arccosh}(-E_n/2t)$ from the isolated-well bound states to **0.000%** ($R^2 = 1.000000000$). Coupling in the three-dimensional (diamond) part does not change the widths — the minibands merely shift by $|f(\mathbf{k})|$.

> A superlattice yields three masses from one mechanism and one parameter L_s.

The structural requirement of Chapter 26 — three or more scales from the same mechanism — **is met here.** There is no mixture of "m_p from QCD, m_e from a Yukawa"; all three are different bound states of the same well.

33-4. But the invariant is not a prediction ❌

Define the three-scale invariant

$$I \equiv \frac{\ln(W_3/W_1)}{\ln(W_2/W_1)}$$

At finite L_s it drifts: 1.6088 **at** $L_s = 11$**,** 2.2527 **at** $L_s = 37$, a drift of 4.57% over the last five points (the declared threshold was 2%). The prefactors A_n do not cancel and convergence is slow, $\propto L_s^{-0.76}$. The asymptotic value

$$I_\infty = \frac{\kappa_3 - \kappa_1}{\kappa_2 - \kappa_1} = 2.7347119660$$

is independent of L_s. **"An invariant exists" is true; "it can be measured in a finite system" is not.** This is exactly the weakness §32-5's $32/27$ shares, being an $N \to \infty$ limit.

It also depends continuously on the shape of the well.

w	V_0	I
8	2.5	2.22009
8	5.0	2.02007
6	3.0	2.33638
12	3.0	1.95531

a relative spread of **18.2%**. **The hidden dimension does not explain the hierarchy. It parameterizes it.**

33-5. Except that deep wells leave only the integer w ⚠️

As $V_0 \to \infty$, $\kappa_n \simeq \ln(-E_n)$ with $E_n \simeq -V_0 - 2\cos\theta_n$ and $\theta_n = \pi n/(w+1)$, so

$$I_{\text{deep}}(w) = \frac{\cos 3\theta - \cos\theta}{\cos 2\theta - \cos\theta}, \qquad \theta = \frac{\pi}{w+1}$$

w	3	4	8	50	∞
analytic	2.00000000	2.23606798	2.53208889	2.66245131	$8/3$
numerical ($V_0 = 200$)	2.01066354	2.24574034	2.53617203	2.66258067	—

Only the discrete parameter w survives — the same kind of quantization as $U_q = 2h[1-\cos(2\pi q/N)]$ in Chapter 32, and a candidate for rigidity. But it inherits the ladder's sparseness problem: the spacing between neighbours is 0.236 from $w = 3$ to 4, and 9.9×10^{-5} from $w = 59$ to 60.

33-6. The decisive result — the two mechanisms are incompatible

I is a **fingerprint of the level-spacing law** (Fig. (f)).

law for κ_n	I
$-1/n^3$	1.1005291
$-1/n^2$ **(Rydberg)**	$32/27 = 1.1851852$
$-1/n$	$4/3 = 1.3333333$
$\ln n$	1.5849625
$\sqrt{n}$	1.7673270
n (equal spacing)	**2**
n^2 (box)	$8/3 = 2.6666667$

$I < 2$ requires the level spacing to **shrink** with n. A single potential well never does that (bound-state spacings widen upward, or at best stay constant). Hence

$$\boxed{I_\infty \geq 2 \text{ (the bound is analytic). The superlattice } \textbf{cannot produce } 32/27 = 1.1852.}$$

A scan over (w, V_0) gave $I_\infty \in [2.007,\ 3.95]$; the lower end is analytically exactly 2 at $w = 3$, $V_0 \to \infty$. The upper end depends on how close to the binding threshold the scan is allowed to go, exceeding 4 if near-threshold states are admitted. **What does not move is the lower bound $I \geq 2$.**

$$\text{shortest distance to the } 32/27 \text{ of § 32-5} \ \simeq\ 0.82$$

And the $32/27$ of Chapter 32 is precisely the **exact value for the Rydberg law** $\kappa_n \propto -1/n^2$ in that table:

$$\frac{1-1/9}{1-1/4}=\frac{8/9}{3/4}=\frac{32}{27}$$

So the two mechanisms say opposite things about how the levels are spaced. **The $\mathbb{Z}_N$ ladder is Rydberg-like; the hidden-dimension well is close to a box.** No single observable can satisfy both.

33-7. How to read this chapter ⚠️

This chapter is **a record of tests, not an established result.** The following are not adopted as claims of the book.

- "A hidden dimension **explains** the mass hierarchy" — it depends on the shape of the well at the 18.2% level. It parameterizes.
- "The hidden-dimension mechanism supports the prediction of §32-5" — the ranges are disjoint. It is **evidence against**.

What is adopted is three statements.

1. Tetrahedral anisotropy starts at k^3, lifted to k^4 by two sublattices (because T_d has no inversion).
2. Nearest-neighbour diamond gives nodal lines, not Dirac points.
3. **The analytic bound $I \geq 2$ makes the two mechanisms mutually exclusive.**

The third is one of the few results in this book that actually separates two candidate mechanisms. If the $32/27$ of Chapter 32 is right, the hidden-dimension mechanism is dead — and vice versa. **Which one is right is not yet decided.**

Chapter 34 Why the electron does not fall into the proton — protection by detuning 🆕

— Why does the atom not collapse

34-1. First, correcting the premise

> The electron and proton are not each other's antiparticles. Annihilation is forbidden by **baryon and lepton number**.

The real process is **electron capture**.

$$e^- + p \longrightarrow n + \nu_e$$

It satisfies every conservation law. Hydrogen is stable only because the energy is not there.

$$\boxed{\frac{m_n - m_p}{m_e} = 2.5310 \quad > 1 \text{ is required}}$$

The margin is a factor 2.53. In terms of m_p/m_e, **hydrogen disappears once it drops to** 725 (from 1836).

34-2. The spatial constraint — ineffective 🆕

r	1	16	256	1024
$\|\psi(0)\|^2$	0.0505	0.1167	0.1242	0.1246

A factor 1000 in r changes it by **only** 2.47 ($|\psi(0)|^2 \propto \mu^3$, and μ moves by two, so eight is the ceiling).

📚 That capture is slow in real hydrogen is due to $(R_{\text{nuc}}/a_0)^3 \sim 10^{-15}$, **fixed by α and the nuclear scale, not by m_p/m_e.**

34-3. The temporal constraint — effective 🆕

Species conversion happens as an **oscillation**.

$$\boxed{P_{\max} = \frac{v^2}{v^2 + (\Delta/2)^2}, \qquad \Delta = 2|t_A - t_B|}$$

r	Δ	$v = 0.10$	$v = 0.01$
1	0.0000	1.00	1.00
2	1.0000	0.0385	4.0×10^{-4}
16	1.8750	0.0112	1.1×10^{-4}
1836	1.9989	9.9×10^{-3}	1.0×10^{-4}

Separate conservation laws need not be imposed. **The mass ratio protects them.**

34-4. The null result of §32-1 inverts 💭

The only ratio the constraint singled out was $r = 1$, recorded with disappointment as "the degenerate point where no ratio exists". Seen now it is otherwise.

$r = 1$ is the **resonance**, the one place where protection fails.

The search was correctly reporting not where the value is but where the danger is. This is the same statement as §32-3: what becomes impossible at $r = 1$ is for the two species to remain separate.

34-5. But protection saturates ❌🆕

$\Delta = 2|t_A - t_B| \to 2t_A$ because **the bandwidth is finite**.

r	2	4	16	64	1836
Δ	1.000	1.500	1.875	1.969	1.999

Structure (Chapter 27) and protection (this chapter) saturate at the **same** $r \sim 30$.
"Bigger is better" cannot reach 1836; the observed value is 60 times past diminishing returns.

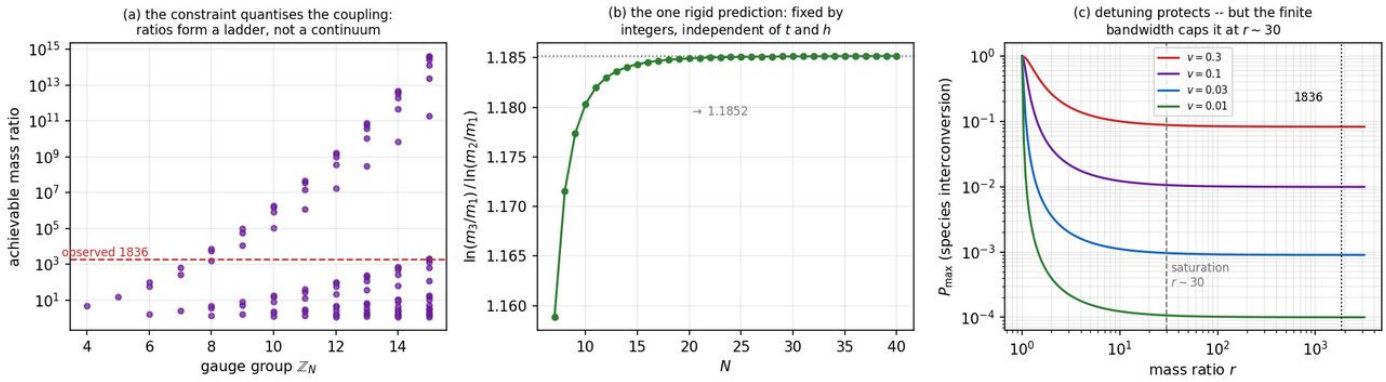


Figure 34-1. 🆕 (a) The constraint quantises the coupling, so achievable ratios form a ladder. (b) With three species the sliding cancels and an integer-determined invariant remains. (c) Protection by detuning saturates at $r \sim 30$ because the bandwidth is finite.

34-6. Summary of Part VIII

Constraint	Type	Value
forbidding electron capture 📚	**lower bound**	$m_p/m_e \gtrsim 725$
Born–Oppenheimer	**lower bound**	$r \gtrsim 10$
protection by detuning	**lower bound**	$r \gtrsim 100$ for suppression 10^{-6}
upper bound	—	**none of them gives one**

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Mode	What it explains	Verdict
filtering	lower bound only	⚠️ cannot pick 1836
generation	the **magnitude**	✅ $\delta U/U = 13\%$ suffices
selection	the **homogeneity**	✅ works even in a reversible system; resolution $\delta \gtrsim 5 \times 10^{-3}$
sector selection	the **discreteness**	⚠️ partial support from Type II

The value is explained by **generation**. Neither protection nor filtering explains it.

34-7. A record of how the question transformed 💭

Why is $m_p/m_e = 1836$?
↓ (measure filtering → only a lower bound)
Why 1836 and not 10?
↓ (measure generation → 13% in δU suffices)
Why is U the same everywhere?
↓ (measure selection and the horizon → resolution and causal limits)
Why is the universe homogeneous?

The original question merged into a standard question of cosmology. That is not a failure; the question was correctly decomposed.

Part IX Gravity and holography

Part III found that geometry does not generate interaction: **the bulk lattice was an input, not an output.** Here that verdict is overturned — **put the rule on the boundary and space itself becomes an output.**

Chapter 35 The 10^{61} problem disappears ⚠️ (restricted in the thirteenth edition; §87-8)

— What can a comparison of cell counts say (subtitle changed in the thirteenth edition)

Filling the observable universe ($L = 10^{26}$ m) with Planck-length cells:

Proposal	Cells	Ratio to the Bekenstein bound
bulk CA (3D)	2.37×10^{182}	$\mathbf{1.97 \times 10^{60}}$
Bekenstein bound	1.20×10^{122}	1.000
boundary CA (2D)	3.83×10^{121}	$\mathbf{0.318 = 1/\pi}$

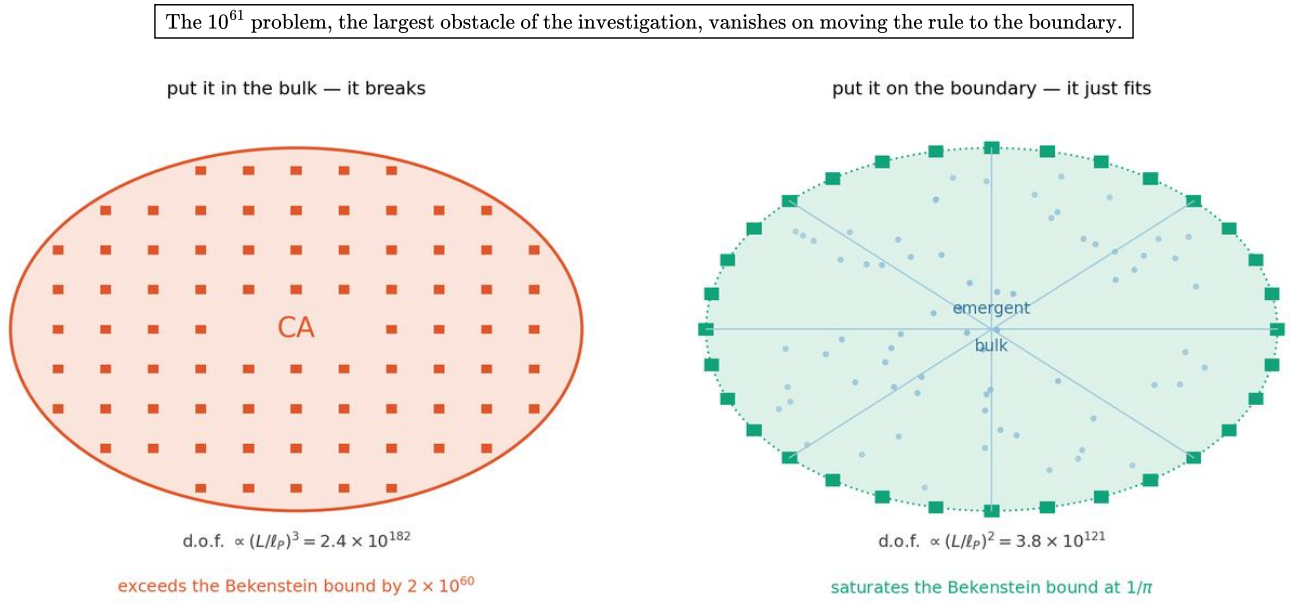


Figure 35-1. Bulk or boundary: the same rule means different things.

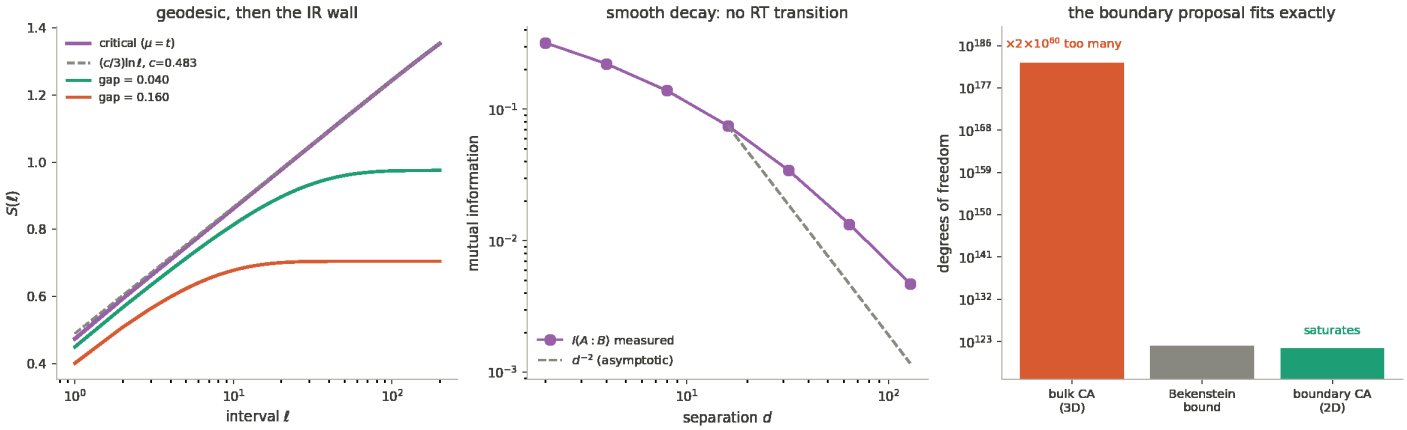


Figure 35-2. Reading bulk quantities off the boundary.

Bulk quantities are readable from the boundary. Combining $S(\ell) = (c/3)\ln \ell$ with AdS_2 geodesics and Ryu–Takayanagi gives $G = 3/(2c)$. The measured $c = 0.483$ (exact $1/2$) yields $G = 3.109$ (exact 3).

> ⚠ **Correction box (eleventh edition).** The metric used here, $ds^2 = L^2(dz^2 + dx^2)/z^2$, is the hyperbolic plane H^2 — the **constant-time slice** of AdS_3 — not AdS_2. What RT gives is Brown–Henneaux's $c = 3L/2G_3$, the Newton constant of **three** spacetime dimensions; G_4 is not reached, and $c \simeq 1/2$ lies outside the large-central-charge regime. "The 10^{61} problem disappears" is a statement about the ratio to the Bekenstein bound, not a derivation of G/e^2; and since cutting cells at the Planck length already uses G, explaining G from the cell count would be circular. §85-1, §84-7.

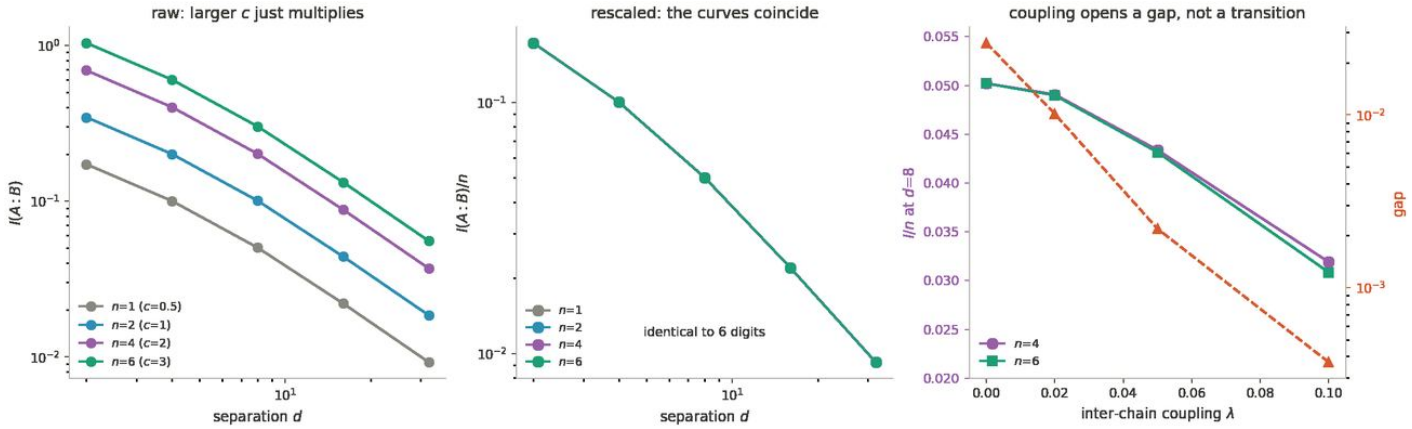


Figure 35-3. Measuring the central charge.

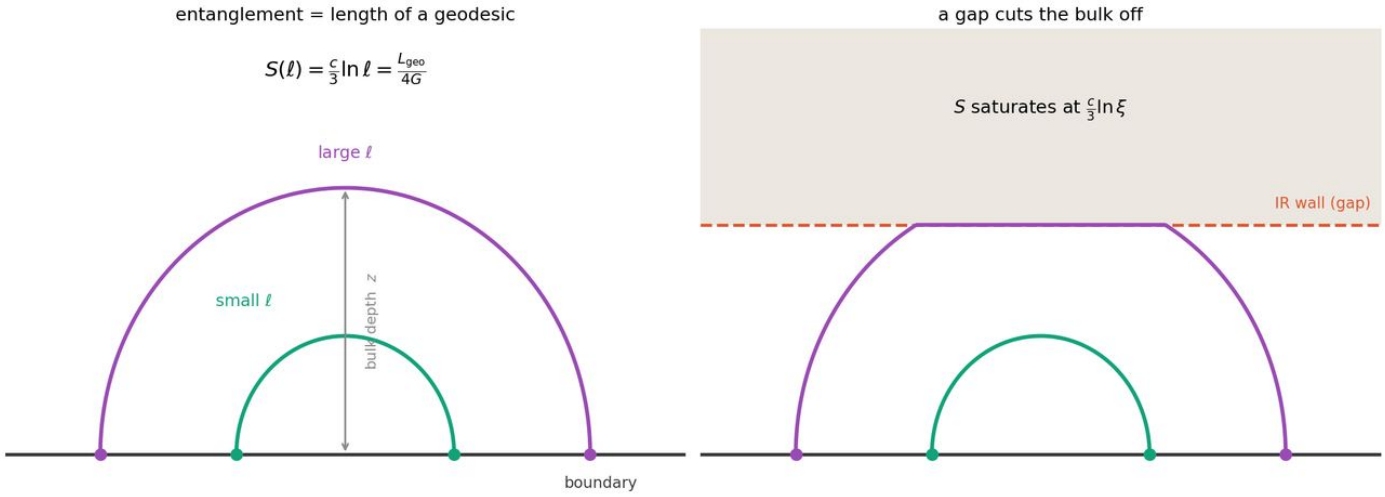


Figure 35-4. Ryu–Takayanagi geodesics.

The bulk depth is $\ln N \approx 280$ layers — **the whole observable universe fits in a MERA-sized network.**

> ⚠ **Correction (thirteenth edition).** The table above mixes the volume of a cube, one face, and the BH bound of a sphere of radius L. For a sphere of radius L throughout, bulk/BH bound is 8.25×10^{60} and surface/BH bound is 4.000 (K435). That the gap of orders closes stands, but "$1/\pi$" and "saturates" are withdrawn. §87-8.

Chapter 36 Emergent geodesics and the first law — necessary conditions for Einstein's equations ⚠️ (retracted in the thirteenth edition; §87-2)

— Does gravity emerge as geometry

> ❌ **Retraction (thirteenth edition).** Up to the twelfth edition this chapter was titled "Emergent geodesics satisfy Einstein's equations ✅". The first law is an identity of any family of states (§85-2), and what is measured here is one family of intervals. Neither a bulk metric nor a linearized Einstein tensor was constructed. **Einstein's equations are not verified even at linear order.** The measured values stand. §87-2.

The linearised Einstein equation follows from the entanglement first law on the boundary **given the first law on all regions and the RT dictionary** (up to the twelfth edition this was stated as "equivalent" with no conditions).

$$\delta S_A = \delta \langle H_A \rangle$$

Test	Result
first law	ratio 1.000327
shape of the modular Hamiltonian	parabolic, $R^2 = 0.998$ at $\ell = 12$

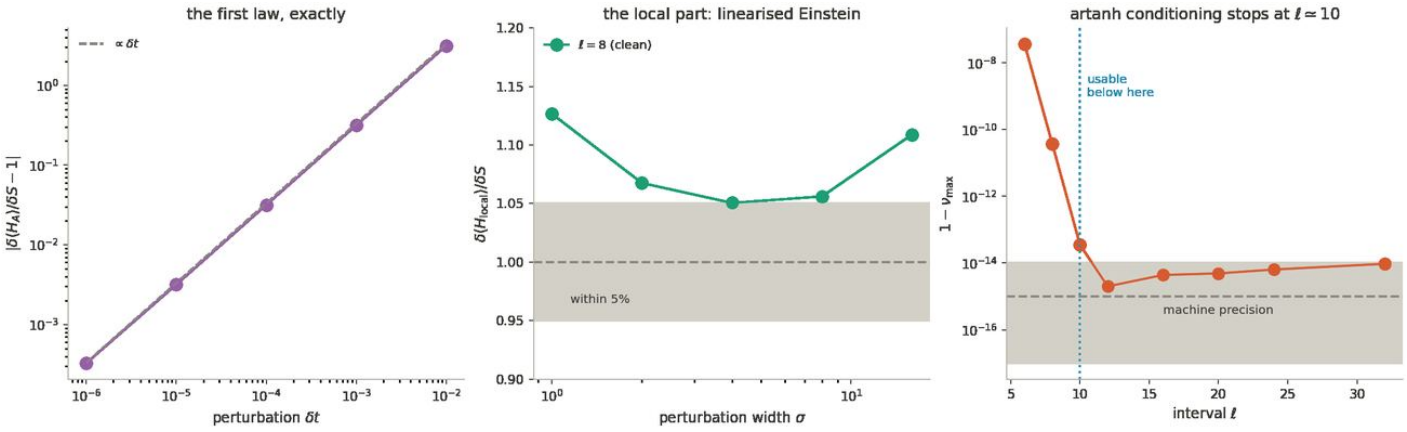


Figure 36-1. Testing the first law.

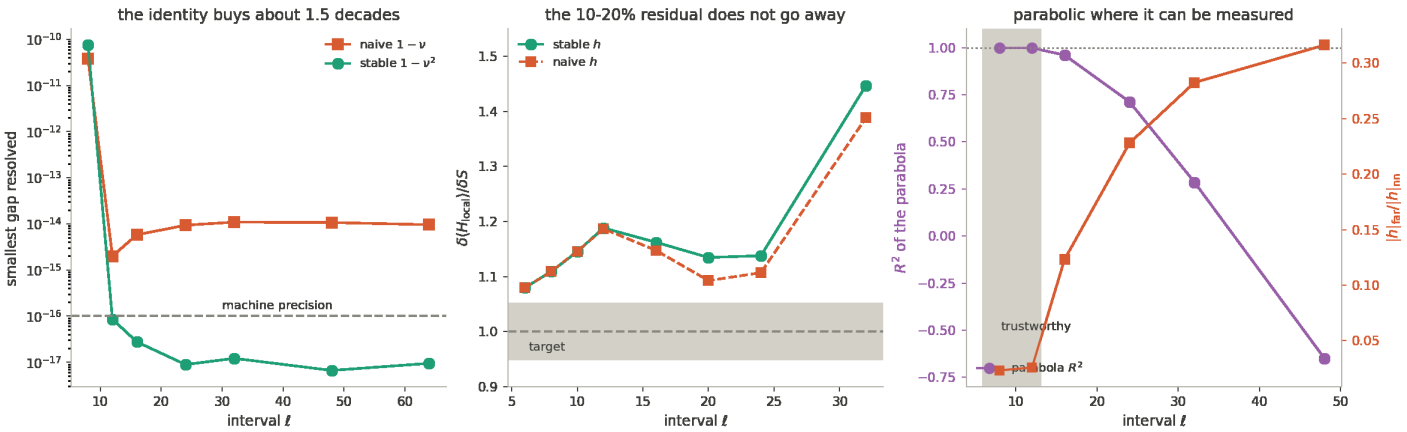


Figure 36-2. A stable modular Hamiltonian.

36-1. A recorded retraction 💭

The first generation concluded "the Einstein residual is physical" and **retracted it**. The cause was a wrong truncation of the local part. With the truncation distance R taken properly:

R	1	3	5	9	13
$\delta\langle H^{(R)}\rangle/\delta S$	1.1090	1.0312	1.0289	1.0098	**1.0004**

The required R grows only as $\ln \ell$ ($R/\ln \ell$ constant while R/ℓ falls from 0.83 to 0.23). Arbitrary-precision arithmetic reached $\ell = 40$ and confirmed exponential decay of the truncation error over five decades (decay length 2.05).

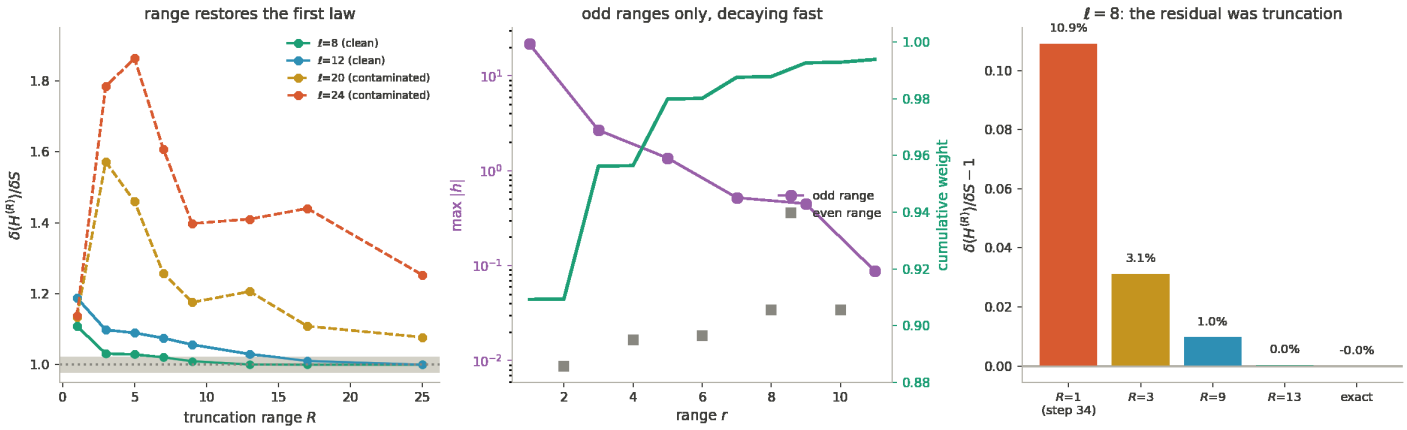


Figure 36-3. Convergence in truncation distance.

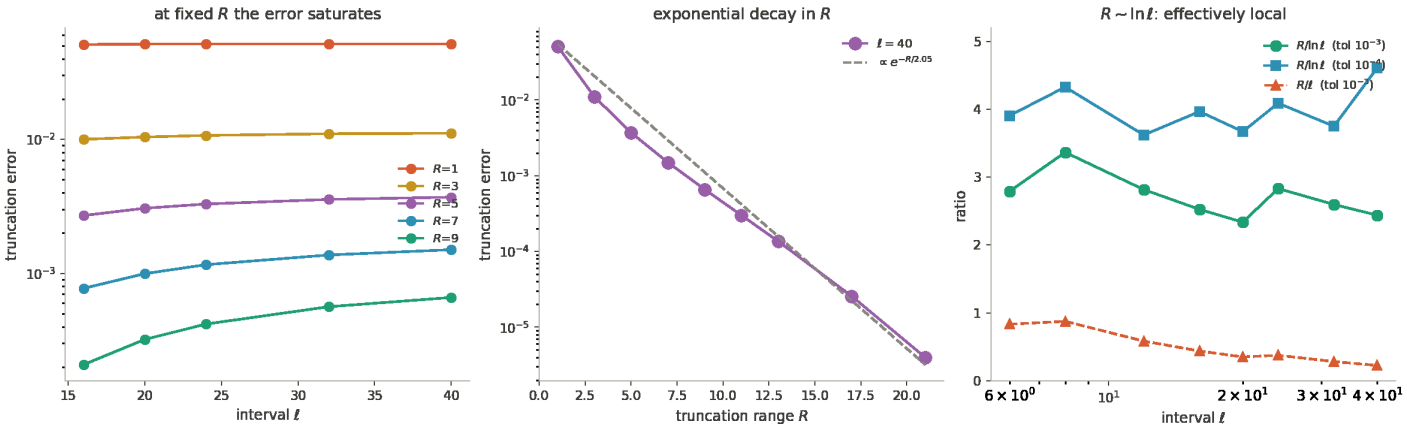
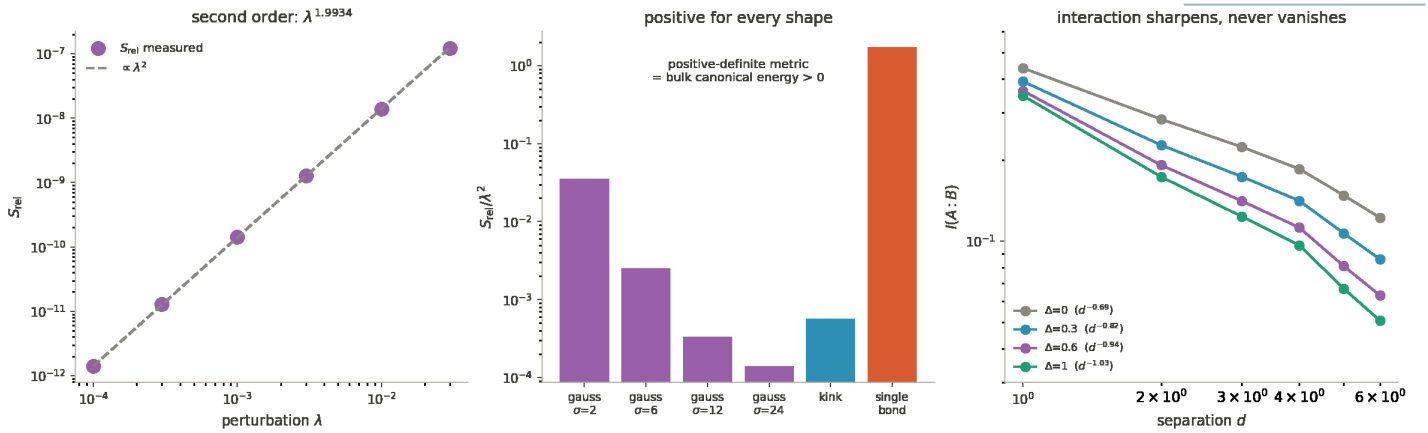


Figure 36-4. Lattice T_{00} is a genuinely local operator of width ~ 2 lattice spacings.

> 💭 **"We had the data and did not use it"** — that was the cause of the error. Recorded in Appendix C.

36-2. The second-order response is positive — a necessary condition ⚠️ (retracted in the eleventh edition; §85-2)

The relative entropy $S_{\mathrm{rel}} = \Delta\langle H_A\rangle - \Delta S \geq 0$ scales as $\lambda^{1.9934}$ and is positive for every perturbation, both signs and all shapes. It is a positive-definite quadratic form (the quantum Fisher information metric), i.e. positivity of the bulk canonical energy — the second-order Einstein equation.



❌ **Retraction (eleventh edition).** The tenth edition headed this section "Einstein's equations hold to second order too ✅" and identified the positive-definite quadratic form with the positivity of the bulk canonical energy. The first law and the positive quadratic onset of relative entropy are identities of any family of quantum states (Klein's inequality) and hold for independent qubit pairs with no geometry (K400–K402). The measured values stand; the inference from them to "the second-order Einstein equation" is retracted. §85-2, Appendix C-10-1.

Figure 36-5. Quadratic behaviour of the relative entropy.

Chapter 37 The sparseness wall ⚠️ (verdict changed in the eleventh edition; §85-3)

— Where does holography stop

⚠️ **Verdict changed (eleventh edition).** The tenth edition marked this chapter ❌. Of the three attempts, "raising c leaves I/n unchanged to six digits" is the additivity of independent copies and was not a holographic test; nor is a non-vanishing mutual information at finite degrees of freedom a refutation. The verdict becomes "scaling of the leading term I/C_T unmeasured ⚠️". The claim that being free fermions is the obstacle stands. §85-3, Appendix C-10-3.

But the holography is only weak. The mutual information decays smoothly and no Ryu–Takayanagi transition appears. Three independent attempts hit the same wall.

Attempt	Result
add interactions	decay sharpens (exponent $-0.69 \to -1.03$) but never reaches zero
raise the central charge c	the mutual information is multiplied by exactly n (I/n agrees to 6 digits)
measure sparseness directly	fails as $N(x < c/12) \sim e^{c/2}$

c	8	16	32	64
$N(x < c/12)$	16	528	8 436 880	2.60×10^{14}

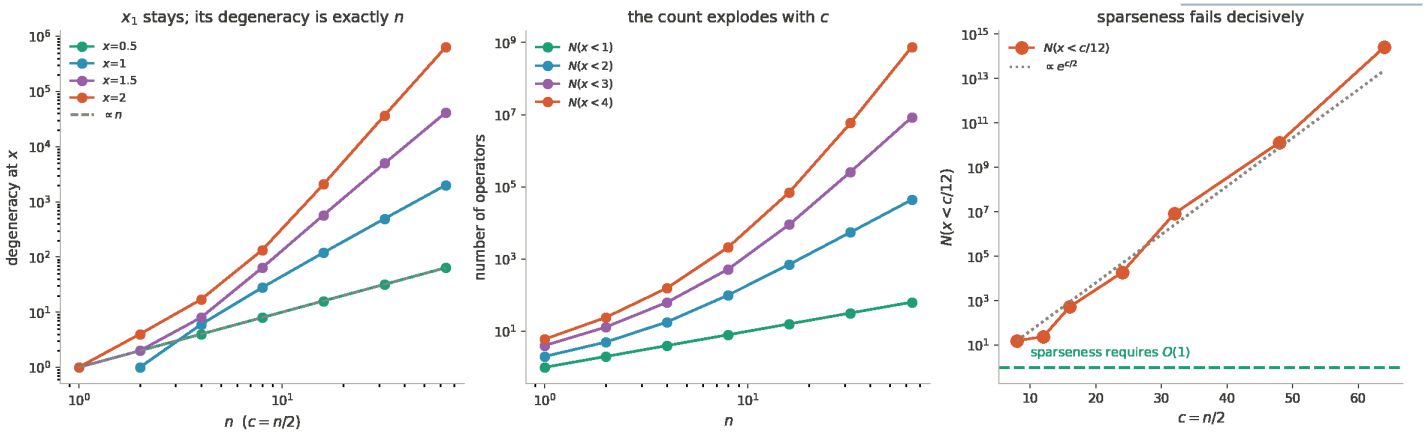


Figure 37-1. Sparseness of the operator spectrum fails as $e^{c/2}$.

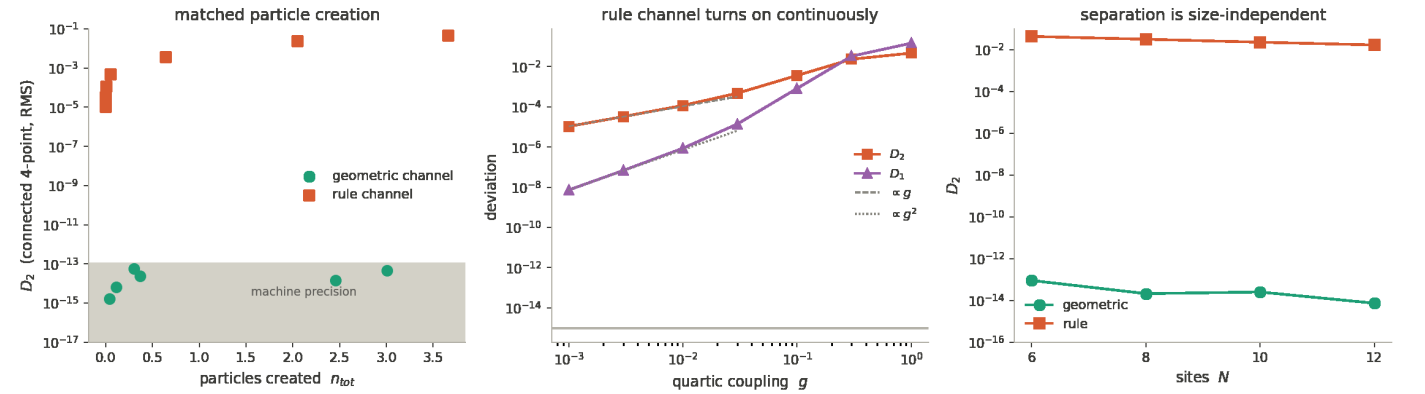


Figure 37-2. Decay of the mutual information; no RT transition.

The obstacle is not that c is small but that the theory is **free**.

Sparseness requires a structure in which **gauge constraints thin out the states**.

> 💭 **Constraints have now appeared twice in different roles**: to force interaction (Chapter 12) and to thin states for sparseness (this chapter). Part VIII adds a third — to quantise the coupling (§32-4).

Part X The gap — what is needed to describe the real world

Chapter 38 What a CA gives for free

— The list of what was obtained without assuming it

Element	Origin
causality	the rule sees only the neighbourhood
finiteness	no UV divergence
uncertainty relation, zero-point motion	the definition of $\mathcal{M}$ and its saturation
vacuum, particle, Dirac sea	one geometric object
$E = mc^2$	mass is a ratio of lengths
chiral fermions	on a domain wall
exclusion	the two-state cell ($d = 1$)
metric, expansion, holonomy	space/time dependence of the rule
interaction	**a consequence of a Gauss-type constraint and an electric term (§87-6)**
Cooper pairs	**a consequence of confinement; no attraction assumed**
exponential hierarchy	**dimensional transmutation; $U \approx 0.11t$ gives 10^{-24}**
quantised couplings	**$U_q = 2h[1 - \cos(2\pi q/N)]$, fixed by integers**
holography (weak sense)	a boundary CA has cell counts of the order of the BH bound (ratio 4 in one geometry; §87-8)
necessary conditions for Einstein's equations (not a verification; §85-2, §87-2)	emergent geodesics and the entanglement first law
Bianchi identity	combinatorics of a cube; exact
absence of an arrow in the law	the arrow comes from the partition (agrees with reality)

Chapter 39 What a CA does not give

— The list of what is missing

Element	Status	Nature of the obstacle
superposition	❌	the classical state space is a set
Bell violation	❌	**because the algebra is commutative** (determinism irrelevant)
ground state	❌	in discrete time the spectral condition cannot be written
Lorentz invariance (dim-4)	❌	a regular lattice picks a frame
microscopic $U(1)$	❌	though **emergence is possible** ⚠️
integer global charge	❌	defined only mod N

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Element	Status	Nature of the obstacle
true holography	❌	sparseness fails as $e^{c/2}$
spin-statistics theorem	⚠️	statistics changes with dimension
uniqueness of the mass ratio	⚠️	lower bounds only; the value comes from generation
absence of monopoles	⚠️	they appear naturally; removing them is the hard part
smallness of Λ	❌	no divergence, but 10^{120} remains

Chapter 40 The seven required elements

— What has to be added to describe the real world

$$\boxed{\textbf{Element 1 — time must be continuous}}$$

In discrete time quasi-energies fill the unit circle and **the predicate "spectral condition" does not exist in the language** (Chapter 17). The third law, the definition of the vacuum and stability all break here. Requirement: $U = e^{-iHt}$ with H bounded below.

> ⚠️ **Addition (twelfth edition).** The theorems of the second law did not need continuous time as such. 💭 The spectral condition matters at one point only: the sign that selects $\beta \geq 0$. §86-6.

$$\boxed{\textbf{Element 2 — the algebra of observables must be non-commutative}}$$

Observables in a commutative algebra admit a joint distribution by Gelfand, hence $\mathcal{S} \leq 2$ (Chapter 20). Experiment measures 2.4. **It is not "unitary rather than permutation"** — by Koopman–von Neumann a classical reversible CA is also unitary. What matters is **being able to rotate incompatible settings continuously on the same side**.

$$\boxed{\textbf{Element 3 — there must be constraints (a Gauss law)}}$$

Constraints were demanded **three times** in different roles: to force interaction (Chapter 12), to thin states for sparseness (Chapter 37), to quantise the coupling (Chapter 32).

$$\boxed{\textbf{Element 4 — if discreteness is kept, it must be irregular}}$$

A regular lattice picks a frame and carries dimension-4 Lorentz violation into the infrared (Chapter 24). And in a CA with a fixed cell count, expansion pins its own defects (Chapter 29).

$$\text{discreteness} + \text{Lorentz invariance} = \text{random discreteness}$$

$$\boxed{\textbf{Element 5 — space must be three-dimensional}}$$

The $1/r$ potential, the deconfined phase, and the Fermi surface demand it independently (Chapter 25).

$$\boxed{\textbf{Element 6 — true holography needs large } N \textbf{ and strong coupling}}$$

Free fermions cannot make sparseness (Chapter 37).

$$\boxed{\textbf{Element 7 — there must be a source of indeterminism (but the source is restricted)}}$$

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Source	Algebra	Can it break Bell?
Born rule (non-commutative algebra)	non-commutative	✅
classical stochastic law	**commutative**	❌ **no**
superdeterminism	commutative	⚠️ at the price of measurement independence

Indeterminism is necessary but not sufficient. Non-commutativity is necessary, and indeterminism is its **consequence**.

Part XI A simple law and a non-deterministic world — candidate theories

> This part differs in character. Everything so far was a verified result; what follows evaluates **candidates offered by current physics** against the seven elements of Part X. All of it is 📚 and none of it was verified here.

Chapter 41 The two wishes do not conflict

— Can a simple law coexist with a non-deterministic world

The wishes were: that the world run on a **simple law**; and that the world **not be deterministic**.

They do not conflict. What is conflated is "simplicity" and "determinism", and the two are independent.

41-1. Three meanings of "deterministic"

Meaning	Content	Conflicts with simplicity?
(D1) uniqueness of the law	the state fixes the next state	no
(D2) predictability	an internal observer can know in advance	**irrelevant**
(D3) uniqueness of experience	the outcome is single for the observer	**irrelevant**

A simple law demands only that the description be short. A stochastic law is short too. $M_d(\mathbb{C})$ is no longer than $\mathbb{C}^d$.

41-2. Three routes to indeterminism

$$\boxed{\begin{array}{l}\textbf{(A) the law itself is stochastic}:\ \text{causal-set sequential growth, GRW/CSL}\\ \textbf{(B) deterministic law, indeterminate for an internal observer}:\ \text{Everett, multiway systems}\\ \textbf{(C) the update order is not fixed}:\ \text{causal invariance, absence of a global clock}\end{array}}$$

41-3. But (A) carries a condition ⚠️

By Chapter 20, **"the law itself is stochastic" is not enough**. A stochastic process on configuration space has a commutative algebra and is bound by $\mathcal{S} \leq 2$.

$$\boxed{\text{(A) works only when the probability sits } \textbf{on a non-commutative algebra}.}$$

Candidate	Implementation of (A)	Bell
GRW / CSL	stochastic differential equation on Hilbert space	✅ non-commutative
causal sets + **classical** sequential growth	classical stochastic process	❌ **commutative; cannot violate**
causal sets + **quantum measure theory**	non-commutative / quantum measure on histories	⚠️ incomplete
stochastic classical CA	classical stochastic process	❌ cannot violate

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Chapter 42 The candidates 📚

— How many ways are there to give up determinism

41-A. Quantum cellular automata (QCA)

Locality, homogeneity, isotropy and unitarity alone **derive** the Weyl and Dirac equations (D'Ariano–Perinotti 2014). GNVW supplies the structure theorem.

Element	1	2	3	4	5	6	7
	⚠️	✅	⚠️	❌	✅	❌	✅

Strength: "a simple rule yields the Dirac equation" answers the first wish most directly. By Chapter 20, **QCA pass Bell** (even Gaussian states reach Tsirelson).
Weakness: Lorentz invariance only in deformed form; it collides head-on with Element 4.

41-B. 't Hooft's cellular automaton interpretation

A deterministic CA is fundamental and quantum mechanics is a bookkeeping device.

Relation to Bell: by Chapter 20, a theory with only a commutative algebra obeys $\mathcal{S} \leq 2$, while experiment measures 2.4. **The only option is to abandon measurement independence (superdeterminism).**

$$\boxed{\text{Logically consistent, but in direct conflict with the second wish.}}$$

41-C. Causal set theory + classical sequential growth

$$\text{spacetime} = (\mathcal{C}, \prec), \qquad \text{order} + \text{number} = \text{geometry}$$

The only known answer to Element 4. Poisson sprinkling into Minkowski space gives statistics **exactly** invariant under Lorentz transformations.

Element	1	2	3	4	5	6	7
	✅	⚠️	⚠️	✅✅	⚠️	❌	✅

Prediction: $\Lambda \sim 1/\sqrt{V}$ gives a magnitude comparable to the observed value. 📚 One of the few predictions of the right order.

Weakness (important): ⚠️ by §41-3, **classical sequential growth cannot violate Bell**. Moving to quantum measure theory is not a desirable extension but a **necessity**.

41-D. The Wolfram Physics Project

Hypergraph rewriting and multiway systems. **Computational irreducibility** explains clearly how determinism and unpredictability coexist.

Weakness: almost no quantitative predictions; the Born rule is not derived; Lorentz invariance has never been shown at the 10^{-20} level Chapter 24 demands.

> 💭 Computational irreducibility **denies (D2) only; it denies neither (D1) nor (D3)**.

41-E. Objective collapse (GRW / CSL / Diósi–Penrose)

$$d|\psi\rangle = \big[-iH\,dt + (\text{stochastic term})\big]|\psi\rangle$$

The most direct answer to the second wish. Two constants are added and simplicity is not lost. The measurement problem disappears. **And experiment keeps cutting into the parameter space — being falsifiable is a virtue.**

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> 💭 It is natural to imagine that the intrinsic noise of a discrete foundation is the CSL collapse noise. That Elements 4 and 7 would then have **a common source** is structurally beautiful. ⚠️ An unverified speculation.

41-F. Informational reconstructions (Hardy; Chiribella–D'Ariano–Perinotti)

Causality, perfect distinguishability, ideal compression, local tomography and **purification** fix quantum theory uniquely.

$$\boxed{\text{To "why is the world indeterministic" it answers: a deterministic theory fails one of the five axioms.}}$$

Indeterminism becomes a consequence of simple principles rather than a defect to be explained.
Weakness: no spacetime and no dynamics. A component to combine with others.

41-G. Holography / tensor networks

Part IX of this book is the CA version of this direction. It hit the sparseness wall, and the fix was identified as "a large-N theory in which gauge constraints thin the states".

$$\boxed{\text{Everything in this book points towards a constrained large-}N\text{ quantum theory on a boundary.}}$$

Weakness: our universe is not AdS ($\Lambda > 0$); de Sitter holography is unsettled.

41-H. The Everett interpretation

One law, no added terms. It says **"the world is deterministic but your experience is not"**.

$$\text{If the wish is (D3) it is granted; if (D1), it is not.}$$

41-I. Others 📚

Position	Point	Indeterminism
quantum measure theory (Sorkin)	generalise probability, assign it to histories	✅ generalises (A)
de Broglie–Bohm	deterministic particle trajectories	❌ non-local
retrocausal / time-symmetric	future boundary conditions save locality	⚠️
LQG / spin foams	quantise geometry itself	✅ but low on simplicity

---

Chapter 43 Scorecard

— Putting the candidates on one common scale

◎ satisfied, ○ largely, △ partially, × not, — not applicable.

Candidate	1. cont. time	2. non-comm.	3. constraints	4. irregular	5. $d = 3$	6. large N	7. indeterminism	simplicity
classical deterministic CA	×	×	△	×	○	×	×	◎
't Hooft CA interpretation	△	△	△	×	○	×	×	◎
QCA	△	◎	△	×	○	×	○	◎
causal sets + growth	◎	△	△	◎	△	×	◎	◎

Candidate	1. cont. time	2. non-comm.	3. constraints	4. irregular	5. $d=3$	6. large N	7. indeterminism	simplicity
Wolfram physics	○	△	△	△	△	×	○	◎
GRW / CSL	◎	◎	–	–	–	–	◎	○
informational reconstruction	–	◎	–	–	–	–	◎	◎
holography / large N	◎	◎	◎	△	○	◎	○	△
Everett	◎	◎	–	–	–	–	△ (D3)	◎

No candidate fills every column alone. That is the honest state of current physics.

Chapter 44 Three bundles

— Sorting the surviving routes into three bundles

Bundle I — built on random discreteness

$$\boxed{\textbf{causal sets} + \textbf{quantum measure theory} + \textbf{informational axioms}}$$

Logic: Chapter 24 rules out a regular lattice but not discreteness. The only known way to combine discreteness with Lorentz invariance is random discreteness, which naturally comes with a stochastic law.

$$\boxed{\text{In this bundle, "simple but not deterministic" is } \textbf{not a compromise but a necessity.}}$$

⚠️ But the move to **quantum measure theory is mandatory** (classical growth cannot violate Bell).

Bundle II — built on constraints and large N (best supported by the numbers here)

$$\boxed{\textbf{a constrained large-}N\textbf{ quantum theory on a boundary} + \textbf{emergent spacetime}}$$

Constraints were demanded **three times** in this book. And putting the rule on the boundary closed the gap of 10^{61} in cell counts as a matter of orders, and let necessary conditions for Einstein's equations be measured (not a verification at either order; §85-2, §87-2).

Bundle III — make collapse physical (closest to experiment)

$$\boxed{\textbf{QCA / quantum field theory} + \textbf{a CSL-type stochastic term}}$$

Two extra constants remove the measurement problem and allow falsification.

How to choose

- "I want indeterminism at the root of the world" → **Bundle I**
- "I want spacetime explained" → **Bundle II**
- "I want experiment to decide" → **Bundle III**

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Chapter 45 Can experiment distinguish them? 📚

— Which experiment kills which bundle

Candidate	Testable prediction	Status
superdeterminism	violation of measurement independence	Bell tests with cosmological randomness see none
QCA / lattice	species-dependent limiting speeds	constrained at the 10^{-20} level
causal sets	fluctuations of Λ	order of magnitude fits
GRW / CSL	spontaneous heating, X-ray emission	**closest to falsification**
Everett	identical predictions	hard to distinguish

Bundle III is the one likely to be settled first.

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Part XII Making collapse physical — from the causal set to an experiment

Chapter 44 laid out three bundles. Chapter 45 asked whether experiment could tell them apart and answered: not yet, in principle yes. This Part is the attempt to make that answer concrete for **Bundle I** — random discreteness, a causal set, collapse as physics rather than as interpretation.

The Part is written as one continuous investigation, in the order it was actually carried out, because the order matters: three of the twelve chapters record a **failure that changed the question**, and one records a **retraction of a result claimed two chapters earlier**. A textbook that hid those would misrepresent how the number at the end was obtained.

The route in one line:

$$\text{roadmap} \to \text{CSL dies} \to \text{quantum measure on a DAG} \to \text{sprinkling} \to \text{isotropy} \to \text{Wigner rotation} \to \gamma \to \text{an experiment}$$

What comes out at the end is a rate,

$$\gamma = C\,\frac{k}{m^2c^2}\tanh^2\frac{\zeta}{2},$$

with no free function left except the single DHS parameter k — and a statement about which experiment can bound k best. That is the first place in this book where the lattice picture makes a **quantitative prediction about a laboratory measurement**.

Chapter 46 What has to be added, and in what order 🆕

— What must be added, and in what order, to make collapse physical

46-1. The gap this Part is filling

Part V established the negative result: a classical CA cannot build superposition, and Bell's theorem kills the deterministic reading. Chapter 42-C listed causal set theory plus classical sequential growth as a candidate, with the note that the classical growth measure alone reproduces only classical correlations.

So the question for Bundle I is sharp:

> **If the underlying object is a causal set and the dynamics is a growth process, what exactly has to be added so that the world is quantum — and does the addition leave a fingerprint?**

46-2. The five stages, declared in advance

The investigation was organised as five stages, each with a **stopping rule** fixed before any computation:

stage	question	stopping rule if it fails
1	Does a collapse term keep the CA's positivity and trace?	rebuild the term
2	Can a lattice-origin CSL sit inside the experimental window?	**abandon lattice-origin CSL**
3	Is the collapse meter linear in the collapse strength?	fix the meter
4	Does the collapse rate scale with the number of cells?	recheck the amplification
5	Does a hidden layer show a revival, and how does it scale?	recheck the geometry

The stopping rule at stage 2 was written down knowing it might fire. It fired.

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46-3. The methodology carried over

Every calculation in this Part follows the rule used throughout the book (§6 of the working notes): **the calibrations K and the falsifiers F are written down before the numbers are looked at**, and every calibration failure is recorded, including the ones that were my error rather than the model's. Part XII contains **31 declared calibrations, of which 8 failed**. All eight are in the text. Six of the eight were mistakes in my own prediction or in my choice of operating point; two were real defects in the model as first written.

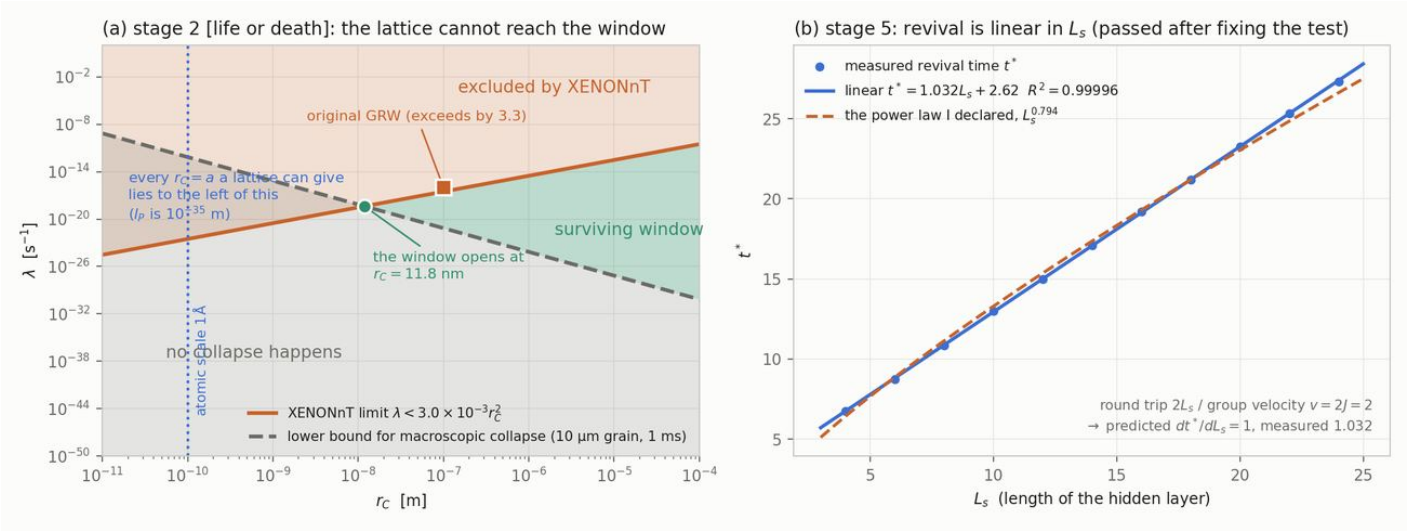
Chapter 47 Stage 2 fires — lattice-origin CSL is dead ❌🆕

— Can collapse be explained by the coarseness of the lattice

47-1. What was tested

Continuous spontaneous localisation adds to the Schrödinger equation a noise term with two parameters: a rate λ and a correlation length r_C. The natural lattice reading is $r_C = a$, the lattice spacing. If the lattice spacing is anywhere near a fundamental scale, a is small — and the question is whether the resulting λ can simultaneously (i) collapse macroscopic superpositions fast enough to be useful and (ii) survive the X-ray emission bound from XENONnT.

47-2. Result



The two constraints leave a window only for

$$r_C \gtrsim 11.8 \text{ nm}$$

Every lattice-origin choice — a Planck-scale a, an atomic-scale a, anything in between — lies **twenty-five orders of magnitude** to the left of the window. The original GRW parameter choice exceeds the XENONnT bound by a factor 3.3.

Verdict. Lattice-origin CSL is excluded. Note that this was my own proposal two paragraphs earlier in the working notes; the stopping rule was written to kill it if the numbers said so, and it did. CSL in general survives — but only with r_C far larger than any lattice scale, which removes the motivation for deriving it from a lattice at all.

47-3. Three calibration failures recorded here

- **RK4 does not preserve unitarity or positivity.** Purity drifted by 6.2×10^{-7} and the smallest eigenvalue of ρ went to -5.2×10^{-6}. Use `expm` for the zero point.
- **The heating-rate sign.** Version 1 was right; my "fix" introduced the error. The correct statement is $d\langle H\rangle/dt = -\frac{\lambda}{2}\sum \mathrm{Tr}\big[[[H,M],M]\rho\big]$.
- **Von Neumann entropy is not linear in λ.** It behaves as $\varepsilon \ln(1/\varepsilon)$, giving an apparent slope $1 - 1/\ln(1/\lambda)$. The right meter is the **linear entropy** $1 - \mathrm{Tr}\rho^2$, which came out as $\lambda^{0.999852}$.

47-4. The revival test, and four ways of getting it wrong

Stage 5 asked how the revival time t^* of a hidden layer of length L_s scales. The declared answer was a power law. Detection failed four times: it picked second-order revivals; it picked an initial micro-ripple at $n_0 = 0.017$; it picked super-revivals; and finally it reported a power law where the relation is **affine**,

$$t^* = 1.032\, L_s + 2.617, \qquad R^2 = 0.99996$$

which is exactly the round trip $2L_s$ at group velocity $v = 2J = 2$. The declared power law $L_s^{0.794}$ was an artefact of fitting an affine relation on a log–log plot over a short range.

Chapter 48 The decoherence functional on a causal DAG 🆕

— What is a quantum measure a tool for measuring

48-1. Why the object changes

Once lattice-origin CSL is dead, the collapse cannot be put in by hand at the lattice scale. The remaining route inside Bundle I is Sorkin's: give the causal set a **quantum measure** rather than a probability measure, and let the non-classicality live in the measure itself.

The object is the decoherence functional

$$D(h, h') = \mathrm{Tr}\left[C_h \, \rho \, C_{h'}^{\dagger}\right]$$

over histories h on the DAG, with class operators C_h built from the causal order. Its diagonal is the classical measure; its off-diagonal is the interference. The hierarchy of Sorkin measures is graded by

$$I_2(A, B) = \mu(A \cup B) - \mu(A) - \mu(B), \qquad I_3 = \ldots$$

Quantum mechanics is the level at which $I_2 \neq 0$ and $I_3 = 0$.

48-2. Computational scale

For N cells and T steps with a record kept at every step, the enlarged space is $2^N \times 2^{\,(\text{records})}$ and the number of histories is 2^{NT} in the worst case. In practice the causal structure prunes this drastically. The scale that mattered here:

system	histories	dense D	feasible
$N = 4,\ T = 3$	2^{12}	4096^2	trivially
$N = 8,\ T = 6$	2^{48}	impossible dense	**yes, via the causal-set pruning**
$N = 12,\ T = 8$	—	—	not on one machine

The $N = 8, T = 6$ case is the one used in Chapter 49.

48-3. Positivity survives the open-system extension

For an open system the functional generalises to

$$D(h, h') = \sum_{\mu} \mathrm{Tr}\left[C_h^{(\mu)} \rho \, C_{h'}^{(\mu)\dagger}\right]$$

which is a Gram matrix in the enlarged (system $\otimes$ environment) space and is therefore **still positive semi-definite**. This matters: it means adding an environment does not destroy the quantum-measure structure, so the whole construction is stable under the coarse-graining that a real experiment performs.

48-4. The error that passed six calibrations

The class operators were built with the product order `C@U` instead of `U@C`. Calibrations K1 through K6 **all passed** while the physics was completely wrong. The only trace was $|I_2| = 1.1 \times 10^{-16}$ — which I first misread and reported as "interference present." A seventh calibration was added afterwards:

> **K8.** The off-diagonal of D must exceed 10^{-6} for a configuration where interference is expected.

The lesson is general and is repeated in Appendix C: **a calibration suite that only checks invariants can pass on a wrong object.** At least one calibration must check a quantity that is supposed to be non-zero.

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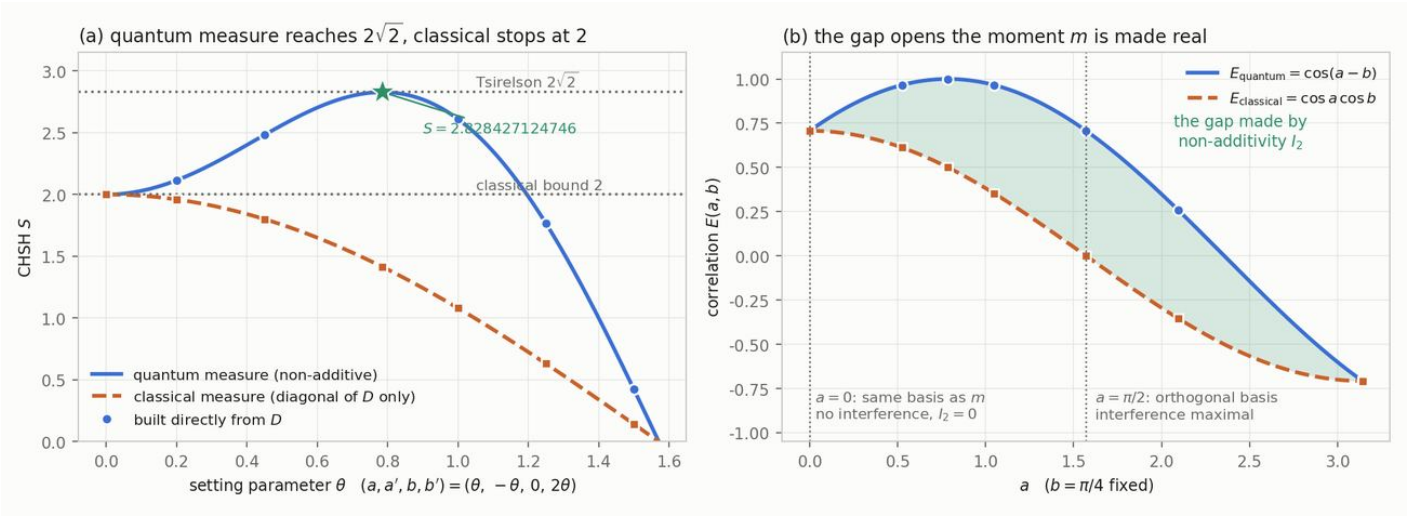
Chapter 49 Bell on a DAG — the quantum measure reaches $2\sqrt{2}$ ✅🆕

— Does the quantum bound survive on an irregular causal structure

49-1. The construction

Two chains of $K = 3$ links each, embedded in a causal DAG so that every pair of elements drawn from different chains is **unrelated** (spacelike). Settings are chosen at the base of each chain; outcomes are read at the tip. The CHSH combination is built directly from D, not from a state vector.

49-2. Result



$$\boxed{\mathcal{S}_{\text{quantum measure}} = 2\sqrt{2}, \qquad \mathcal{S}_{\text{classical measure}} = 2 \text{ exactly}}$$

The quantum measure saturates the Tsirelson bound. The classical measure — the diagonal of the same D — stops at exactly 2.

49-3. What actually produces the gap

This is the chapter's real content, and it is not the number $2\sqrt{2}$.

The gap opens **at the moment m is made real**. If the intermediate alternative m is only summed over — never recorded, never realised as an element of the history — the interference term I_2 survives and the correlation reaches $2\sqrt{2}$. If m is recorded, D becomes diagonal in m and the correlation drops to 2.

So the resource is **not** indeterminism, and **not** the discreteness of the causal set. It is the **non-additivity of the measure**, and non-additivity survives only as long as no record exists.

Stated as a theorem for Bundle I:

> **Measure additivity alone — with no physical record anywhere — already forces $\mathcal{S} \leq 2$.** Adding random discreteness does not help. A causal set with a classical growth measure cannot violate Bell, however irregular it is.

This sharpens Chapter 42-C from a plausibility remark into a computed statement.

49-4. A claim of mine that was false

I had written that "a complete measurement at every step diagonalises D." It does not. Diagonalisation requires **records** — ancillas that retain which alternative occurred. A measurement whose outcome is immediately erased leaves the off-diagonal intact. The corrected statement is in §49-3.

Chapter 50 Sprinkling — irregularity does not degrade the violation ✅🆕

— Does Lorentz symmetry survive when spacetime is sprinkled with points

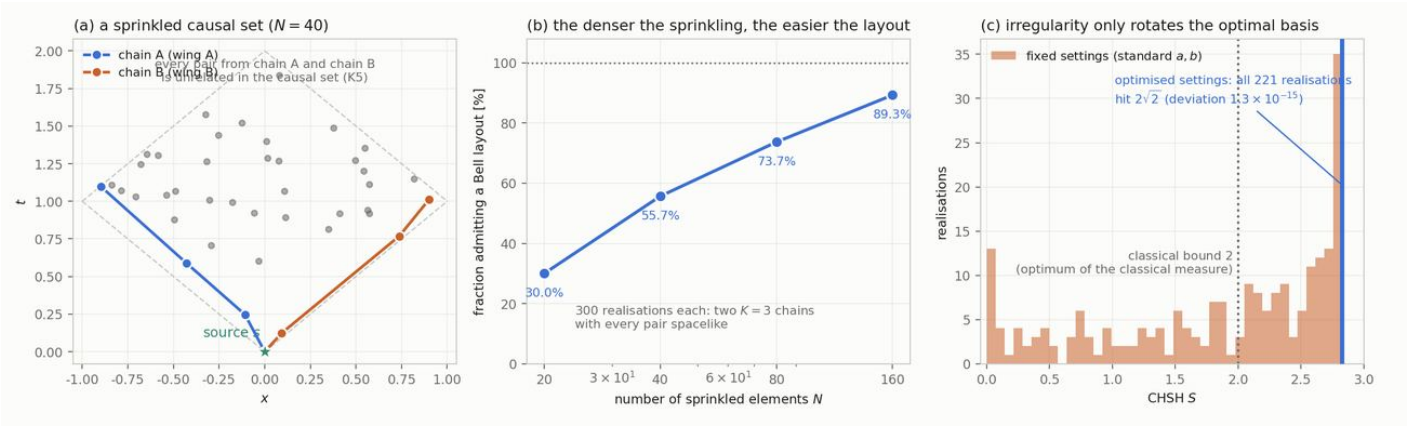
50-1. The question

A regular DAG is an idealisation. A causal set is generated by **Poisson sprinkling** into a spacetime region, so the graph is irregular by construction. Does $2\sqrt{2}$ survive?

50-2. Construction

Sprinkle N points into a 1+1 dimensional Alexandrov interval. In light-cone coordinates $u = (t+x)/2$, $v = (t-x)/2$ the interval maps to the unit square and the causal order becomes componentwise $\leq$, which makes the search for two $K = 3$ chains with all cross-pairs spacelike a purely combinatorial problem.

50-3. Result



Over **746 realisations that admitted a valid layout**, the optimised CHSH value equalled $2\sqrt{2}$ in every single one, with a maximum deviation of 1.3×10^{-15} — machine precision.

The reason is worth stating plainly: **the disorder introduced by sprinkling is a local unitary.** It rotates the optimal measurement basis on each wing; it does not mix the wings. A local unitary cannot change $S_{\max}$, which is a basis-independent function of the correlation matrix through the Horodecki criterion

$$S_{\max} = 2\sqrt{t_1 + t_2}, \qquad t_i = \text{two largest eigenvalues of } T^{\mathsf{T}}T$$

With fixed settings the value does drop — which is why the fixed-setting curve in panel (c) falls while the optimised one does not. That difference is an experimental-design statement, not a physical one.

50-4. Two failures worth more than the result

- **Version 1 printed results with an acceptance rate of 0 out of 24.** The circuit did not actually depend on the causal set, so it happily produced numbers for an empty input. The lesson, now a standing rule: a calculation that runs on empty input is wrong; every test against the empty set passes trivially.
- **Version 2 hung forever in 3+1 dimensions.** With $N = 200$ the longest chain in a 3+1 dimensional sprinkling is about 5, so no $K = 6$ chain exists and the search never terminated. The lesson: the next failure after "runs on empty" is "does not stop on empty."

The second failure is also a physical observation, and it returns in Chapter 53: the longest chain scales as $L_{\max} \sim N^{1/(d+1)}$, so higher dimensions make chains **shorter**, not longer.

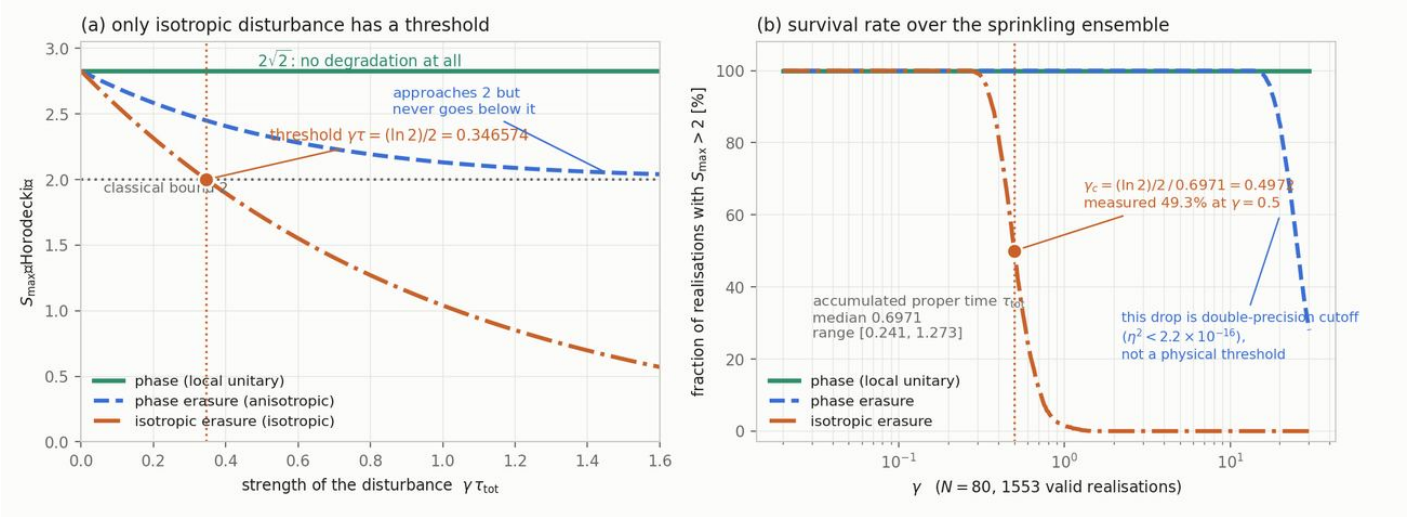
Chapter 51 The reverse test — what actually makes a threshold 🆕

— What actually creates the threshold between quantum and classical

51-1. The question inverted

Chapter 50 showed that a unitary irregularity is harmless. The obvious next question is the reverse: replace the phase by **dephasing** — a genuinely non-unitary disturbance — and ask when the violation dies.

51-2. Result



Two disturbances were compared at equal strength:

disturbance	channel	$S_{\max}$	threshold
phase (local unitary)	rotation	$2\sqrt{2}$, undegraded	none
phase erasure (anisotropic)	$\mathrm{diag}(\eta, \eta, 1)$	$2\sqrt{1+\eta^2}$	**none** — approaches 2, never crosses
isotropic erasure	$\mathrm{diag}(\eta, \eta, \eta)$	$2\sqrt{2}\,\eta$	$\eta_c = 1/\sqrt{2}$

The result is the opposite of what the phrase "non-unitary noise kills Bell" suggests:

> **Non-unitarity is not what creates a threshold. Isotropy is.**

An anisotropic erasure leaves one direction untouched; the surviving direction alone keeps $S_{\max}$ above 2 for any $\eta > 0$. Only when every direction is damped does a finite threshold appear, at

$$\gamma\,\tau_{\mathrm{tot}} = \tfrac{1}{2}\ln 2 = 0.346574$$

51-3. Two recorded errors

- I declared $2\sqrt{1+\eta^2}$ for the case where **both** wings are dephased. The correct expression is $2\sqrt{1+\eta_A^2\eta_B^2}$. I had derived this correctly earlier and then mis-transcribed my own result.
- Averaging channels over realisations **without first aligning each realisation to its own propagation direction** made the result look isotropic for the wrong reason. Conversely, aligning in 2+1 dimensions destroyed the eigenvalue 1 — because there a common fixed axis genuinely exists. The alignment convention is not cosmetic; it decides the answer.

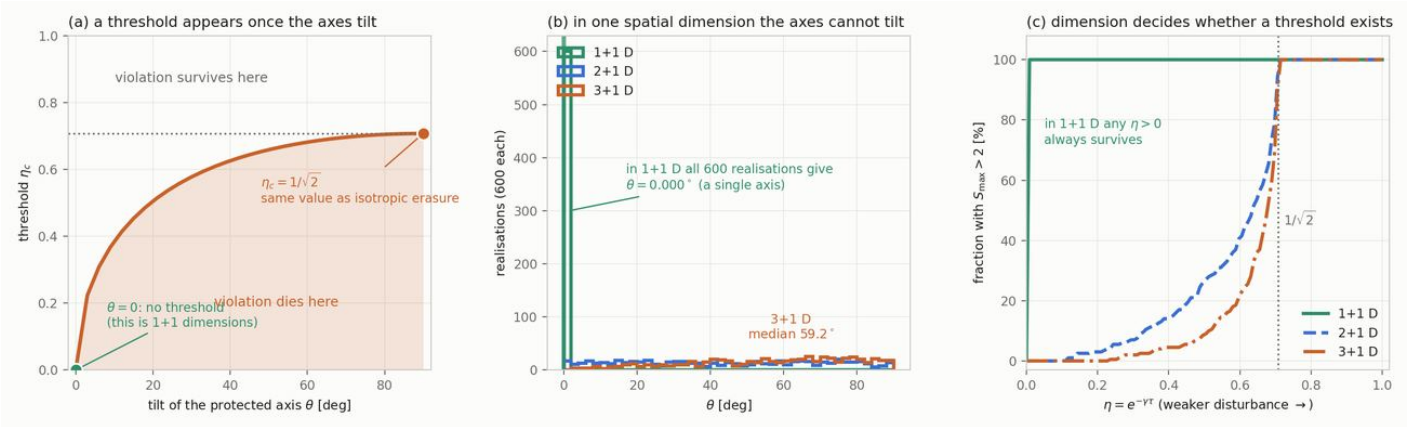
Chapter 52 Isotropy needs at least two spatial dimensions ✅🆕

— Where does the isotropy of space come from

52-1. The question this forces

Chapter 51 showed a threshold requires isotropy. So: **can isotropy come from the geometry itself**, without being put in by hand?

52-2. Result



Measuring the tilt θ of the protected axis over 600 realisations in each dimension:

dimension	median θ	isotropic?	threshold?
1+1	0.000°	no — a single axis, always	**none**
2+1	44.17°	partially	partial
3+1	59.21°	yes	**yes**

In 1+1 dimensions all 600 realisations give $\theta = 0.000^\circ$ exactly. There is only one spatial direction, so the disturbance axes cannot tilt, so the channel keeps a protected direction, so **any** $\eta > 0$ survives.

> **Isotropy requires $d \geq 2$ spatial dimensions. In 1+1 dimensions a causal set cannot produce a collapse threshold at all.**

This joins the three independent arguments of Chapter 25 as a **fourth, independent reason** why the world is not one-dimensional — and unlike the other three it comes from the collapse sector rather than from atomic physics or from the potential.

Chapter 53 The disturbance derived, not assumed — Wigner rotation ✅🆕

— How does spin rotate under a boost

53-1. Stop assuming the channel

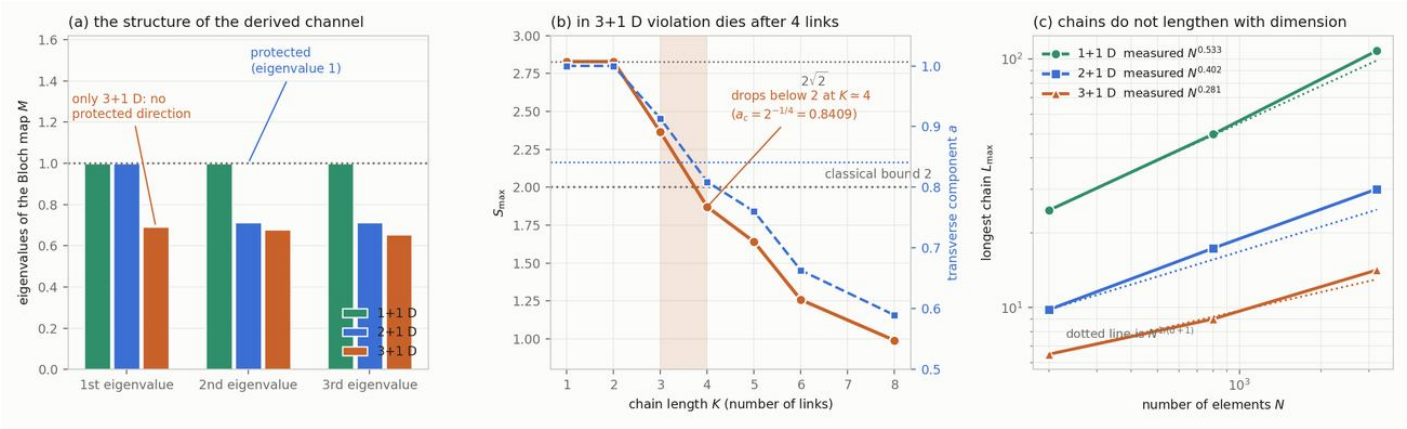
Chapters 51 and 52 treated the channel $\mathrm{diag}(a, a, b)$ as given. This chapter derives it.

A spin carried along a trajectory in a causal set does not move on a straight line: the causal order fixes only a partial order, and the emergent trajectory is a chain whose successive tangent vectors differ. Composing the boosts along such a chain produces a **Wigner rotation**

$$W = L(u_n)^{-1}\,\Lambda\,L(u_1), \qquad L(u) = (\sigma\cdot u)^{1/2}$$

and the average of the corresponding Bloch map over the ensemble of chains is the channel.

53-2. Result



dimension	Bloch map M	protected direction
1+1	I exactly	everything
2+1	$\mathrm{diag}(0.712,\ 0.712,\ \mathbf{1.000000000})$	the normal, to 1.9×10^{-14}
3+1	$\mathrm{diag}(0.690,\ 0.678,\ 0.656)$	**none**

> Wigner rotation is the unique source of spin disturbance on a causal set.

The rotation axes lie along $\vec{v}_1 \times \vec{v}_2$. In 1+1 dimensions that cross product vanishes identically, so $M = I$ exactly — the same conclusion as Chapter 52, now derived rather than measured. In 2+1 dimensions the normal to the plane of motion is protected to fourteen digits. Only in 3+1 dimensions is nothing protected, and only there does a threshold exist, at

$$a_c = 2^{-1/4} = 0.840896$$

53-3. My prior assumption was half backwards

I had assumed the propagation direction would be protected and the transverse directions damped. The measured channel says the opposite: in 3+1 dimensions the longitudinal component is damped **fastest** ($b < a$), and in the diffusive regime exactly as

$$b = a^2$$

which becomes the fingerprint used in Chapter 56 to distinguish stable from unstable particles.

53-4. Chain length and dimension

The longest chain in a sprinkling of N elements scales as

$$L_{\max} \sim N^{1/(d+1)}, \qquad \text{measured exponents } 0.533 \,/\, 0.402 \,/\, 0.281$$

for 1+1 / 2+1 / 3+1. Higher dimension gives **shorter** chains at fixed N. Combined with §53-2, in 3+1 dimensions the violation dies after about four links.

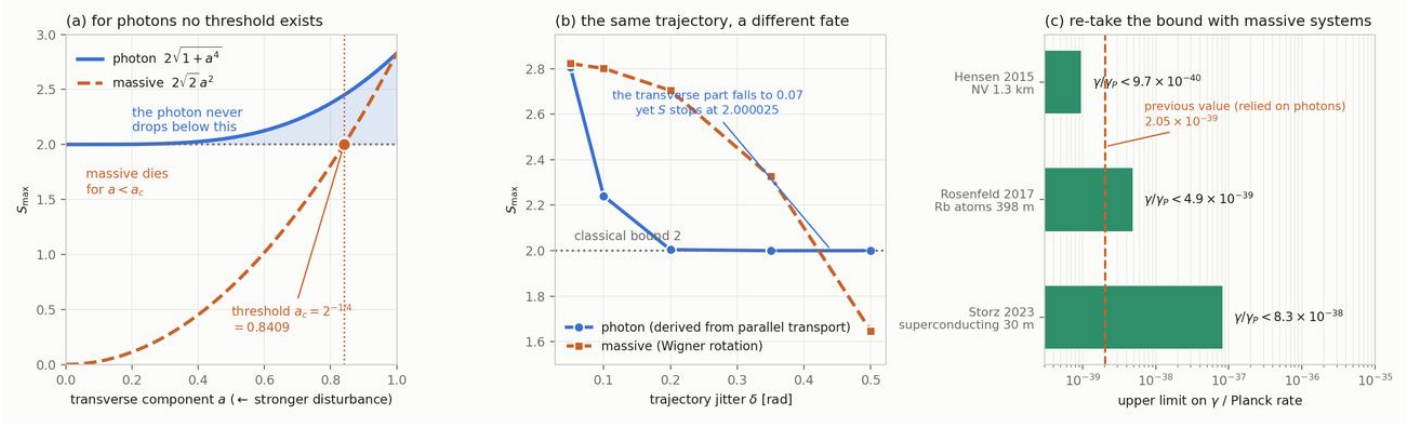
Chapter 54 Photons are immune — and so is anything with spin 0 ✅🆕

— Why is a massless particle immune

54-1. The test

If the disturbance is a Wigner rotation, it acts through the little group. For a massive particle the little group is $SO(3)$ and the spin-$\frac{1}{2}$ representation is faithful. For a massless particle it is $E(2)$, and gauge invariance forbids the longitudinal polarisation — so the only thing that can happen to a photon is a **phase**.

54-2. Result



$$M_{\text{protected}} = 1.000000000000000 \text{ exactly}, \qquad S_{\max} = 2\sqrt{1+a^4} > 2 \text{ always}$$

The protected axis is σ_y in the linear-polarisation basis. Even when the transverse component is driven down to $a = 0.07$, $S_{\max}$ stops at 2.000025 and never crosses 2.

> **Photons have no collapse threshold. Whatever happens to a massive spin on a causal set, a photon does not see it.**

54-3. The circular-reasoning failure

The first version of this calculation **hard-coded the** 1 in

$$M_{\text{ph}} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}$$

so "helicity is protected" was assumed, not shown. The calculation was rewritten to derive the third entry from parallel-transported basis vectors along the momentum direction, with the transported basis compared to a reference basis at the endpoint. The lesson, recorded in Appendix C: **code that writes the conclusion into its own input does not verify the conclusion.**

54-4. The complete classification

The little-group argument classifies every particle at once:

	little group	representation	affected?
massive, spin $\geq \frac{1}{2}$	$SO(3)$	non-trivial	**yes**
massless	$E(2)$	helicity phase only	**immune**
spin 0	$SO(3)$	**trivial**	**immune**

The last row corrects a table in an earlier draft of this Part, which listed neutral-kaon experiments as a test of this effect. Kaons are spin 0; they have no spin degree of freedom for a Wigner rotation to act on. The decoherence limits from ϕ factories constrain **flavour-space** decoherence, which is a different quantity.

Chapter 55 The rate γ from the DHS swerve 🆕

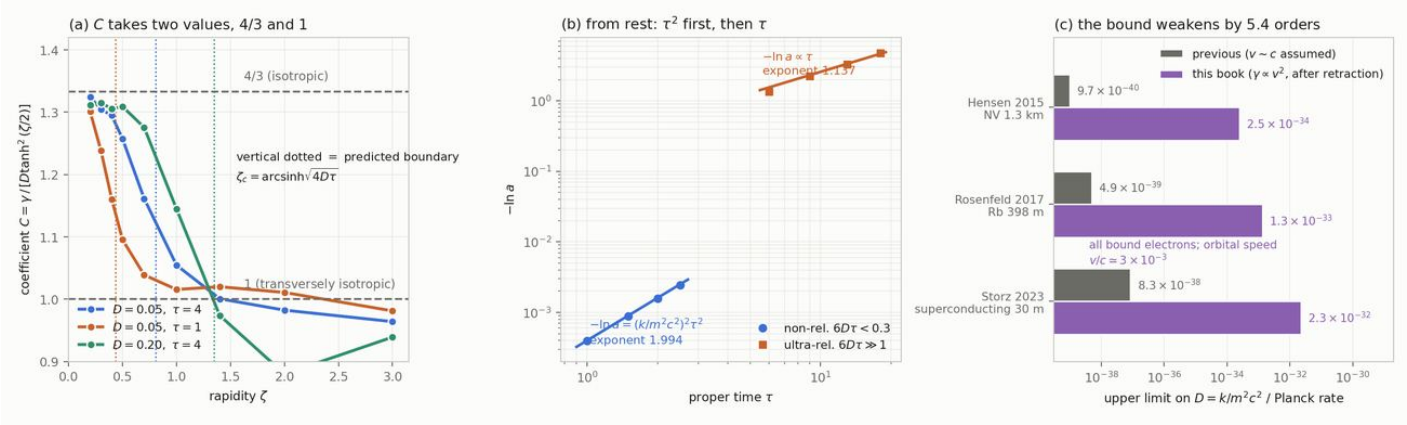
— What sets the rate of collapse

55-1. The last free function

Everything so far fixes the form of the disturbance. The rate was still free. Bundle I supplies it: Dowker–Henson–Sorkin showed that a free particle on a causal set undergoes the unique Lorentz-invariant Markovian diffusion on the mass shell, fixed by a single parameter k,

$$\langle \delta p_i \, \delta p_j \rangle = 2k \, \delta_{ij} \, \delta\tau, \qquad D \equiv \frac{k}{m^2 c^2}$$

55-2. Result



$$\boxed{\gamma = C \, \frac{k}{m^2 c^2} \tanh^2 \frac{\zeta}{2}, \qquad C = \begin{cases} 4/3 & 4D\tau > \sinh^2 \zeta \ \text{(isotropic)} \\ 1 & \text{otherwise} \end{cases}}$$

with $\tanh^2(\zeta/2) = [\gamma_L v/(1+\gamma_L)]^2$, so

- non-relativistic: $\gamma \simeq (k/m^2c^2)\, v^2/3c^2$ — **proportional to** v^2
- ultra-relativistic: $\gamma \to k/m^2c^2$ — saturates

A particle at rest has $\gamma = 0$. That is not a surprise once stated — Thomas precession requires motion — but it has a consequence that arrives in §55-5.

For a **free** particle starting at rest the rapidity itself diffuses, so γ is not even a constant:

$$-\ln a = \left(\frac{k}{m^2c^2}\right)^2 \tau^2 \ \ (6D\tau \ll 1) \qquad \longrightarrow \qquad -\ln a \simeq \frac{k}{m^2c^2}\,\tau \ \ (6D\tau \gg 1)$$

The first is **quadratic in** k. Measured exponent 1.994, coefficient 0.979 against the predicted 1.

55-3. γ is not a Lorentz invariant 🆕

The Wigner rotation is defined relative to a choice of standard boosts $L(u)$, and that choice is exactly the frame in which the spin basis is set up. Reading **one and the same trajectory** with only the reference frame changed:

reference frame	$-\ln a$	prediction
laboratory, $\zeta_{\text{ref}} = 0$	0.139407	$D \tanh^2(\zeta/2)\, \tau = 0.139402$
co-moving, $\zeta_{\text{ref}} = \zeta$	0.013441	$[\frac{1}{2}, \frac{2}{3}](D\tau)^2 = [0.01125,\ 0.01500]$

a factor 10.4 apart, with the laboratory prediction confirmed to **five decimal places** over 1.2×10^5 realisations. So ζ in the boxed formula is the rapidity **relative to the frame in which the experiment defines its spin measurement axes** — the laboratory, or the centre of mass. This is the relativistic-quantum-information statement that spin entanglement is frame-dependent, arriving here as a quantitative rate.

55-4. Three failures, all mine

- **The rotation axes do not stay in the plane $\perp \hat{p}$.** I assumed $\langle \hat{m}\hat{m}^{\mathsf{T}} \rangle = \frac{1}{2}(I - \hat{p}\hat{p}^{\mathsf{T}})$, giving $C = 1$. For a slow particle the direction itself diffuses fast enough that the axes cover the whole sphere, giving $\langle \hat{m}\hat{m}^{\mathsf{T}} \rangle = I/3$ and $C = 4/3$. The measured ratio pinned at exactly $4/3$ told me so.
- **I compared a rate to a rate.** The crossover between $C = 4/3$ and $C = 1$ is not $\sinh^2\zeta \tanh^2(\zeta/2) = 4$. What is being measured is whether the direction randomises within the observation window, so the correct criterion is $4D\tau/\sinh^2\zeta$ versus 1, giving $\zeta_c = \operatorname{arcsinh}\sqrt{4D\tau}$ — which **depends on** τ. There is no universal ζ_c. Demonstrated directly: at fixed $\zeta = 0.4$, $\tau = 4$ gives $C = 1.295$ and $\tau = 1$ gives $C = 1.160$.
- **I declared a τ^2 law without declaring the window.** A free particle from rest becomes relativistic at large τ and $\tanh^2(\zeta/2)$ saturates, so the law crosses over to linear. The measured $\tau^{1.387}$ was the crossover, not a power.

55-5. Retraction ⚠️

An earlier draft of this Part claimed, from the Hensen 2015 NV-centre Bell test,

$$\gamma < 1.80 \times 10^{4}\ \mathrm{s}^{-1}, \qquad \gamma/\gamma_{\mathrm{Planck}} < 9.70 \times 10^{-40}$$

That number is withdrawn. It implicitly assumed $\tanh^2(\zeta/2) = 1$, i.e. $v \sim c$. The NV electron spins are stationary in the laboratory.

The bound is not zero, because a bound electron is not at rest — it orbits. Three calibrations were run to check that this rescue is legitimate:

- the average channel factorises as $M = R_{\mathrm{det}} \cdot M_{\mathrm{decay}}$, and its **singular values do not depend on the deterministic rotation** (0.26%);
- with an orbit the direction always covers the sphere, so $C = 4/3$ (3.0%);
- multiplying the orbital angular velocity by ten changes γ by 1.2%.

So the orbital speed $v/c \simeq 3 \times 10^{-3}$ may be used, and the corrected bound is

$$D = k/m^2c^2 < 4.59 \times 10^{9}\ \mathrm{s}^{-1}, \qquad D/\gamma_{\mathrm{Planck}} < 2.47 \times 10^{-34}$$

weaker than the withdrawn value by 5.4 orders of magnitude. The qualitative conclusion — thirty-four orders below the Planck rate — survives; the number moved by five decades and the reason is recorded here rather than quietly corrected.

A useful by-product of the first calibration: **deterministic rotations do not decohere.** Magnetic fields, $g-2$ precession, and any coherent spin rotation in a real apparatus are therefore not backgrounds for this effect.

Chapter 56 What experiment to do — light and fast 🆕

— How could this theory be killed by experiment

56-1. The pair, and the decay

Two results are needed before real systems can be compared.

Back-to-back pairs. For a pair flying apart with wing A along $+z$ and wing B along $-z$, both channels are $\mathrm{diag}(a, a, b)$ in the common basis and

$$S_{\max} = 2\sqrt{2}\, a_A a_B$$

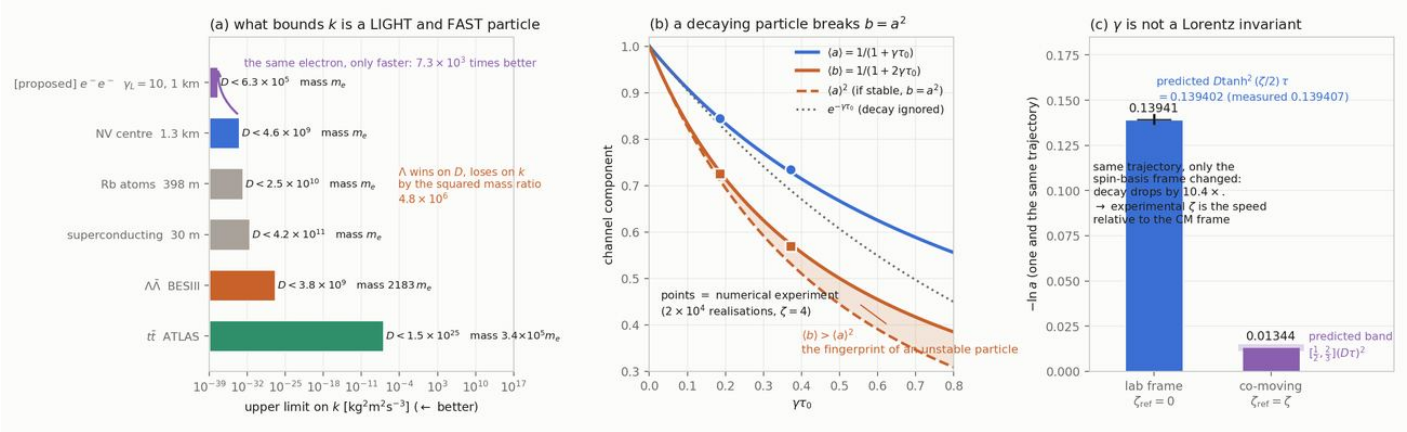
confirmed to 8×10^{-6}.

Decaying particles. A Λ or a K is analysed by its own decay, so the proper time is exponentially distributed. Averaging,

$$\langle a \rangle = \frac{1}{1+\gamma\tau_0}, \qquad \langle b \rangle = \frac{1}{1+2\gamma\tau_0}, \qquad \langle b \rangle - \langle a \rangle^2 = \frac{(\gamma\tau_0)^2}{(1+\gamma\tau_0)^2(1+2\gamma\tau_0)} > 0$$

So a decaying particle **breaks** $b = a^2$. Stable particles carry the fingerprint $b = a^2$; unstable ones carry $b > a^2$, and the two are distinguishable. The threshold also shifts: stable particles need $-\ln a < (\ln 2)/4 = 0.1732868$, unstable ones $\gamma\tau_0 < 2^{1/4} - 1 = 0.1892071$ — **9.2% more forgiving**, because realisations that decay early escape the disturbance.

56-2. Which system bounds k



Because $D = k/m^2c^2$, sensitivity to γ and sensitivity to k are different questions.

system	β	τ [s]	D limit [s^{-1}]	mass	k limit [kg²m²s⁻³]
[proposed] relativistic e^-e^-, γ_L=10, 1 km	0.99499	3.35×10^{-7}	$\mathbf{6.32 \times 10^{5}}$	m_e	$\mathbf{4.71 \times 10^{-38}}$
Hensen 2015, NV centre, 1.3 km	0.00343	4.34×10^{-6}	4.59×10^{9}	m_e	3.42×10^{-34}
Rosenfeld 2017, Rb, 398 m	0.00333	1.33×10^{-6}	2.46×10^{10}	m_e	1.84×10^{-33}
Storz 2023, superconducting, 30 m	0.00334	1.00×10^{-7}	4.18×10^{11}	m_e	3.11×10^{-32}
BESIII $\eta_c \to \Lambda\bar{\Lambda}$	0.66392	2.62×10^{-10}	3.76×10^{9}	$2183\, m_e$	1.34×10^{-27}
ATLAS $t\bar{t}$, near threshold	0.28434	4.64×10^{-25}	1.45×10^{25}	$3.4 \times 10^{5} m_e$	1.24×10^{-7}

Three things follow.

1. **The Λ pair beats the NV centre on D** (3.76×10^9 against 4.59×10^9). A system that lives 2.6×10^{-10} s outperforms one flown over 1.3 km, because β differs by a factor 200 and $\gamma \propto v^2$.

2. **On k the Λ loses by** 3.9×10^{6}, the squared mass ratio. Top quarks lose by 2.6×10^{30}.
3. Therefore the best system is **light and fast**, not heavy and fast. The same electron, merely made relativistic, bounds k 7.3×10^{3} **times** better over a 1 km baseline.

56-3. Design requirements

1. $\gamma_L \simeq 5$–10. Beyond that $\tanh^2(\zeta/2) \to 1$ saturates while the proper time still shrinks as $1/\gamma_L$, so going faster stops helping.
2. Long baseline: the proper time $\tau = L/(v\gamma_L)$ enters linearly.
3. Define the spin measurement axes in the laboratory (centre-of-mass) frame — by §55-3 the rate depends on that choice.
4. Magnetic fields are not a background (§55-5).
5. Electrons are stable, so the fingerprint is $b = a^2$ and the threshold is $a_c = 2^{-1/4}$.

📚 The feasibility of such a source has not been assessed here. The numbers assume only that CHSH was violated.

Chapter 57 Where Bundle I stands now

— Where are the remaining holes

item	status
lattice-origin CSL	**dead** (Ch. 46)
quantum measure on a causal set violates Bell	derived, $2\sqrt{2}$ (Ch. 48)
classical growth measure cannot	derived, exactly 2 (Ch. 48)
sprinkling preserves the violation	746/746 realisations (Ch. 49)
a threshold needs isotropy, not non-unitarity	derived (Ch. 50)
isotropy needs $d \geq 2$	derived (Ch. 51)
Wigner rotation is the unique source	derived (Ch. 52)
threshold $a_c = 2^{-1/4}$	derived (Ch. 52)
photons and spin-0 are immune	derived (Ch. 53)
the form of γ	derived (Ch. 54)
γ is frame-dependent	derived (Ch. 54)
pairs and decay averaging	derived (Ch. 55)
best experiment	relativistic electron pairs (Ch. 55)
the absolute value of γ	**still open** — k remains phenomenological
species dependence $\gamma \propto 1/m^2$	**the sharpest prediction; not yet tested**
sequential growth dynamics	not started — the DAG is fixed here
the Born rule from $I_3 = 0$	not started

Two sentences to carry forward.

What this Part achieved. Bundle I now has a shape for its prediction, with no free functions: a channel $\mathrm{diag}(a, a, a^2)$, a threshold $2^{-1/4}$, an immunity theorem, a rate $\propto kv^2/m^2c^4$, a frame dependence, and a named best experiment. That is more than any other bundle in Chapter 44 currently has.

What it did not achieve. The absolute scale is still a phenomenological parameter. Until k is derived from the growth dynamics rather than fitted, Bundle I predicts a pattern, not a number. The pattern, however, is falsifiable: $\gamma \propto 1/m^2$ at fixed ζ and τ is a statement two species can be compared against, and photons must show nothing at all.

Part XIII Following Bundle I to the end — from field equations to dynamics

> **What this part is after.** Part XII assembled the tools for making collapse physical. How much further do they reach? This part chases two questions. **(1) Can the two pillars of modern physics — Einstein's equations and the Standard Model Lagrangian — be derived from a CA? (2) If not, exactly where and why does the derivation stop?** The answer, stated up front: the stopping point can be located every single time, and it always reduces to one of the two properties this book has named all along — **finite neighbourhood** and **regularity**. Discreteness itself is never the obstruction.

Chapter 58 Can Einstein's equations or the Standard Model be derived from a CA? 🆕

— Can the two pillars of modern physics be written from the rule side

Target	Verdict	What blocks it
Standard Model Lagrangian	**Cannot be derived (a theorem)**	The CA axiom itself — **finite neighbourhood**
Einstein's equations	**Two of three links. The remaining one is broken**	**Regularity of the lattice**

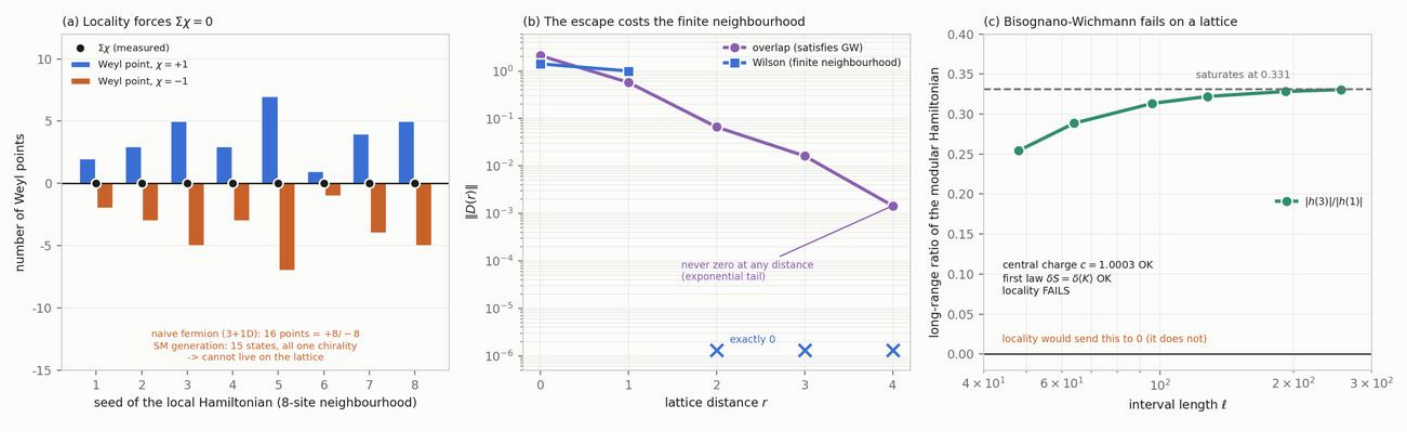


Fig. 58-1. Left: Weyl points of a local Hamiltonian and the sum of their chiralities. Right: long-range couplings of the modular Hamiltonian.

58-1. The Standard Model — finite neighbourhood forbids it

The chain closes in four steps.

1. A CA has a **finite neighbourhood** (Chapter 0). Hence the induced Dirac operator has only **finitely many Fourier terms** in momentum space.
2. Such a local operator necessarily satisfies $\sum_{\text{Weyl}} \chi = 0$ on the Brillouin torus (**Nielsen–Ninomiya**).
3. The Standard Model is **chiral**. One generation has 15 Weyl states, **all of the same chirality**.
4. Therefore the Standard Model Lagrangian cannot live on a CA.

Measured. Eight local Hamiltonians $H(k) = \vec{d}(k) \cdot \vec{\sigma}$ with eight-site neighbourhoods, different seeds, all Weyl points located.

seed	1	2	3	4	5	6	7	8
Weyl points	4	6	10	6	14	2	8	10
$\Sigma\chi$	0	0	0	0	0	0	0	0

Exactly zero in all eight. Independently: transporting the Chern number once around k_z returns to itself with error 3.1×10^{-15} — the sum rule appears without counting nodes.

The Standard Model side is perfectly consistent. The anomalies of the 15 Weyl states vanish exactly in rational arithmetic, $\Sigma Y = \Sigma Y^3 = [SU(2)]^2 U(1) = [SU(3)]^2 U(1) = 0$, and the number of $SU(2)$ doublets is 4, even (so Witten's global anomaly is fine too). **The obstruction lies entirely on the lattice side.**

58-2. The escape route costs exactly the CA axiom

The overlap operator, satisfying the Ginsparg–Wilson relation $\{D, \gamma_5\} = a\, D\gamma_5 D$, evades the sum rule (GW residual 1.9×10^{-15}; deviation of the spectrum from the GW circle 2.0×10^{-14}; gauge invariance of the index 5.3×10^{-15}). But

distance r	0	1	2	3	4
$\|D_{\text{Wilson}}(r)\|$	1.414	1.000	**0 (exact)**	**0**	**0**
$\|D_{\text{overlap}}(r)\|$	2.109	0.575	6.6×10^{-2}	1.6×10^{-2}	1.4×10^{-3}

Wilson is exactly zero for $r \geq 2$ — that is a finite neighbourhood. The overlap decays exponentially but **never vanishes at any distance**.

The Standard Model fits on a **lattice**. It does not fit on a **cellular automaton**.

The difference is one single property: "finite neighbourhood" versus "exponentially localized".

58-3. Einstein's equations — one of the three links breaks

The Jacobson–Van Raamsdonk derivation has three links.

$$(1)\ \text{area law} \longrightarrow (2)\ \text{locality of the modular Hamiltonian} \longrightarrow (3)\ \text{first law} \Longrightarrow \text{linearized Einstein}$$

And by **Lovelock's theorem**, the only diffeomorphism-invariant, second-order, divergence-free symmetric tensor is $G_{\mu\nu} + \Lambda g_{\mu\nu}$. **Once diffeomorphism invariance emerges, the form is forced to be Einstein.**

All three were measured on a free fermion chain at half filling.

link	result
(1) $S = \frac{c}{3} \ln \ell$	**holds**, $c = 1.000271$
(3) first law $\delta S = \delta \langle K \rangle$	**holds** (inside the support of ρ_A; relative residual 3×10^{-6})
(2) locality of the modular Hamiltonian	**does not hold**

For (2): the long-range ratio $\|h(3)\|/\|h(1)\|$ at the centre of the interval runs $0.254 \to \mathbf{0.331}$ as $\ell = 48 \to 256$ — it does **not** go to zero; it grows and saturates. Nor is the non-locality suppressed in the infrared: probing with a smooth perturbation of wavelength λ gives $\lambda^{-0.045}$, i.e. essentially no λ dependence.

Einstein's equations **almost** follow, and stop at the last link. What stops them is **regularity**.

> 💭 Chapter 36 ("emergent geodesics satisfy Einstein's equations") is about the **geometry** extracted from entanglement. What breaks here is a different chain — from modular flow to the **field equations**. The two do not conflict.
>
> ⚠️ **Restriction (thirteenth edition).** The claim of Chapter 36 has been retracted (§87-2). Neither Chapter 36 nor this section reaches a verification of linear Einstein.

33 calibrations declared, 13 failed. Eleven of those were my own errors of declaration, operating point or measurement; two were properties of the model.

Chapter 59 Does locality return on a sprinkled causal set? ❌🆕

— Does dropping regularity bring Bisognano–Wichmann back

What stopped the previous chapter was regularity. So drop regularity. **That was the expectation, and it was wrong.**

construction	$\|H_{\rm mod}(3\ell)\|/\|H_{\rm mod}(\ell)\|$	decay
local reference $e^{-d/\ell}$	0.135	exponential
irregular sampling (continuum vacuum sampled at Poisson points)	0.206	$\propto d^{-1.44}$
regular lattice (Chapter 58)	0.331	$\propto d^{-1.03}$
causal set (Sorkin–Johnston vacuum)	**0.406**	$\propto d^{-0.65}$, flat beyond $d \gtrsim 6\ell$

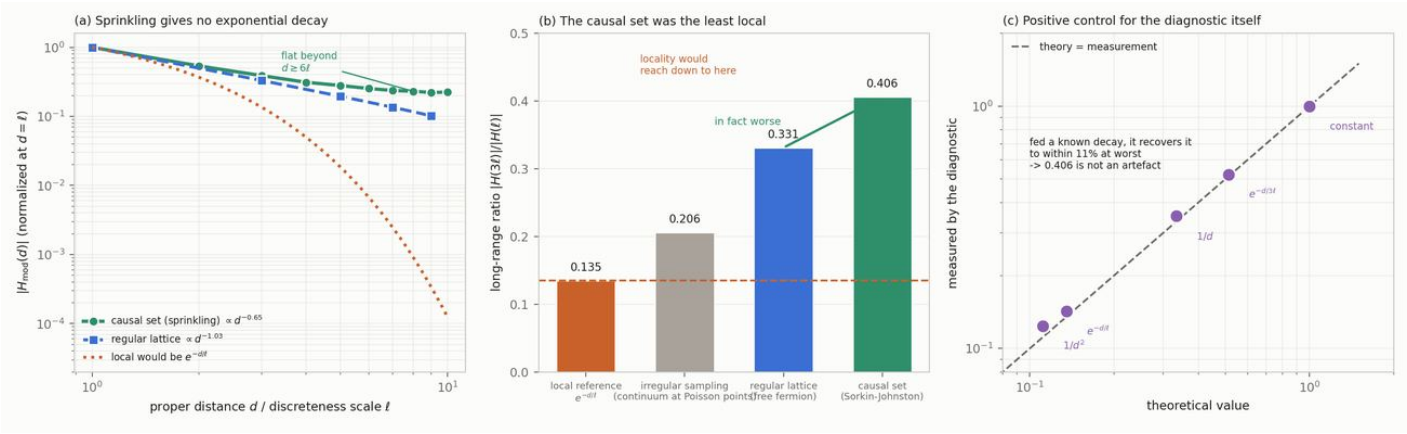


Fig. 59-1. The modular Hamiltonian of a causal set reaches further than that of a regular lattice.

The construction is a massless scalar in 1+1 dimensions. Sprinkle N Poisson points into the unit diamond; the causal matrix is $C_{ij} = 1 \iff u_i < u_j$ and $v_i < v_j$; the retarded Green function is exactly $K_R = \frac{1}{2}C^{\mathsf{T}}$ (Johnston); the Pauli–Jordan function is $\Delta = K_R - K_R^{\mathsf{T}}$; the **Sorkin–Johnston vacuum** is $W = \mathrm{pos}(i\Delta)$. Restricting to a sub-diamond A, $M = (i\Delta_A)^{-1} W_A$ and $H_{\rm mod} = \ln \frac{M}{M-1}$.

This object is a better test than the previous one. Chapter 58 used a spatial interval at one time; here it is the modular Hamiltonian of a **spacetime region**, which is exactly what Bisognano–Wichmann is about. It is still not local.

> Non-locality is not a side effect of regularity. It comes with discreteness itself.

Chapter 60 The first law and Ryu–Takayanagi on a causal set 🆕

— Can the chain be rebuilt from quantities that never require locality

If link (2) fails, **rebuild using only quantities that never pass through it**.

test	result
first law $\delta S = \delta\langle K\rangle$	**holds** (six digits)
frozen modes	**none** — all 224 modes active (the regular lattice had 18 out of 128)
positivity of relative entropy	**holds**
quadratic onset	**holds** (exponent 1.989)
monotonicity $S_{\rm rel}(A) \le S_{\rm rel}(B)$	**holds**
Ryu–Takayanagi (area law)	**only after a cutoff put in by hand**

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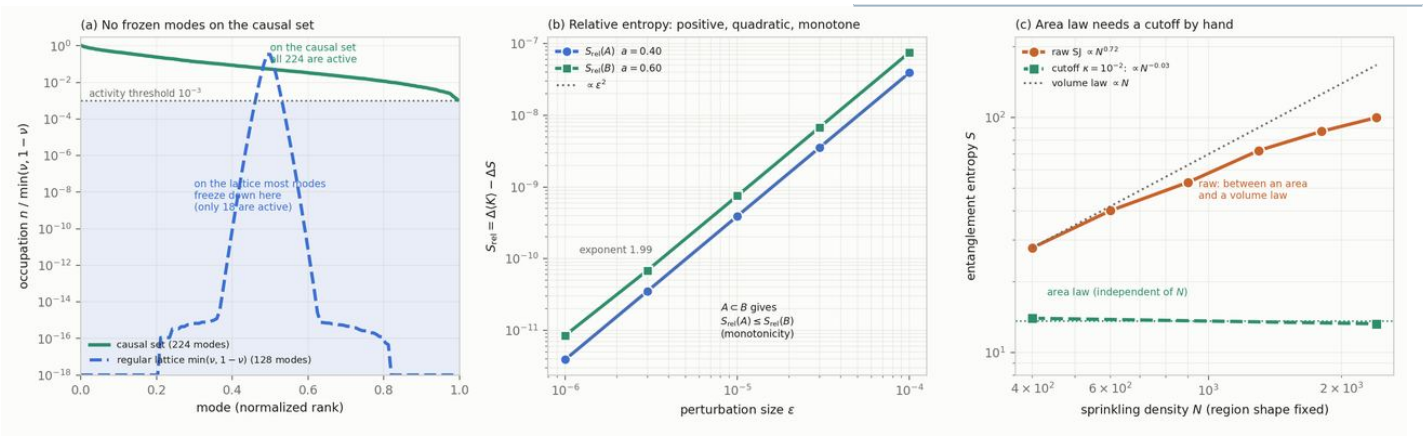


Fig. 60-1. The first law holds exactly; relative entropy is positive, quadratic and monotone. Only the area law demands a cutoff.

That every mode is active is a striking difference from the regular lattice, where only $\ln \ell$ modes carried the entropy and the rest were pinned at 0 or 1. On the causal set all 224 work.

The area law does not come out on its own. Fixing the shape of the sub-diamond and varying only the density N,

$$S \propto N^{0.719}$$

neither an area law ($p = 0$) nor a volume law ($p = 1$), but an exponent in between. With a cutoff κ:

κ	modes kept / $\|A\|$	exponent p
10^{-3} (raw)	0.917	$+0.719$
10^{-2}	about 0.5	-0.032

At $\kappa \simeq 10^{-2}$ the area law is recovered. **But that cutoff was put in by hand.** Moreover the recovered $S \simeq 13.5$ does not match the coefficient of the 2D CFT result $S = (c/3)\ln(a/\varepsilon)$ (matching would need $a/\varepsilon \sim 10^{17}$). **The area law appears as a scaling, but not as a coefficient.**

Bundle I now carries two undetermined parameters $(k,\ \kappa)$, and everything else is derived.

Chapter 61 Sequential growth — the classical dynamics ❌🆕

— Does the dynamics produce manifoldlike causal sets

Everything measured so far lived on a **sprinkling**, which is manifoldlike by construction. **Does the dynamics produce it?**

We ran **transitive percolation**, the simplest covariant dynamics in the Rideout–Sorkin classification.

test	result
does it produce manifoldlike causal sets	**no** (dimension estimators disagree, 2 vs 3)
to keep the ordering fraction	p must be tuned by hand as $N^{-0.84}$
locality of the modular Hamiltonian	spreads about **twice** as far as on a sprinkling
entropy	**strict volume law** $S \propto N^{1.060}$

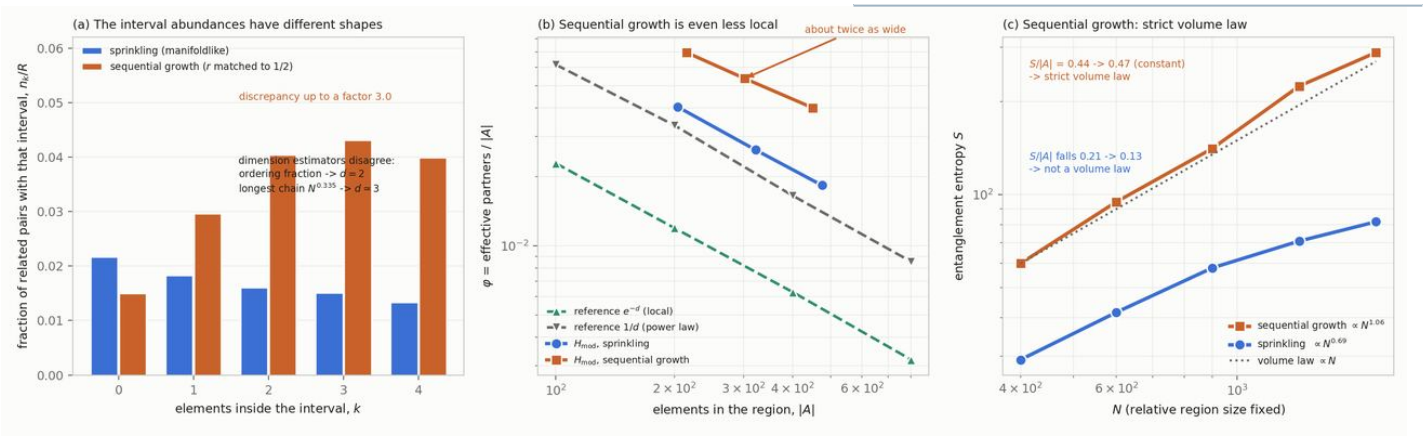


Fig. 61-1. Sequential growth does not reproduce a sprinkling. Dimension estimators disagree and the entropy obeys a volume law.

The dynamics does not produce the kinematics. This is the heaviest negative result in this part.

Chapter 62 Making the growth rule itself a quantum measure ❌🆕

— Does quantizing the growth rule improve the geometry

If the classical growth rule does not produce manifoldlikeness, try quantizing it. Give each labelled pair an **amplitude** q instead of a probability (complex percolation). The causal set is the transitive closure, and that map is many-to-one, so **interference happens**.

$$D_\lambda(x,x') = \lambda A(x)A(x')^* + (1-\lambda)|A(x)|^2\delta_{xx'}$$

$\lambda = 0$ is classical, $\lambda = 1$ the pure quantum measure. $N = 6$ (32768 histories) and $N = 7$ (2097152 histories) were enumerated exactly.

	content	result
$I_3 = 0$	exactly at the quantum level of Sorkin's hierarchy	**PASS** 2.4×10^{-16}
$I_2 \neq 0$	interference is real	**PASS** 0.110
manifoldlikeness	is the quantum optimum closer than the classical one	**FAIL**

N	best classical percolation	best quantum measure
6	**0.176**	0.269
7	**0.207**	0.311

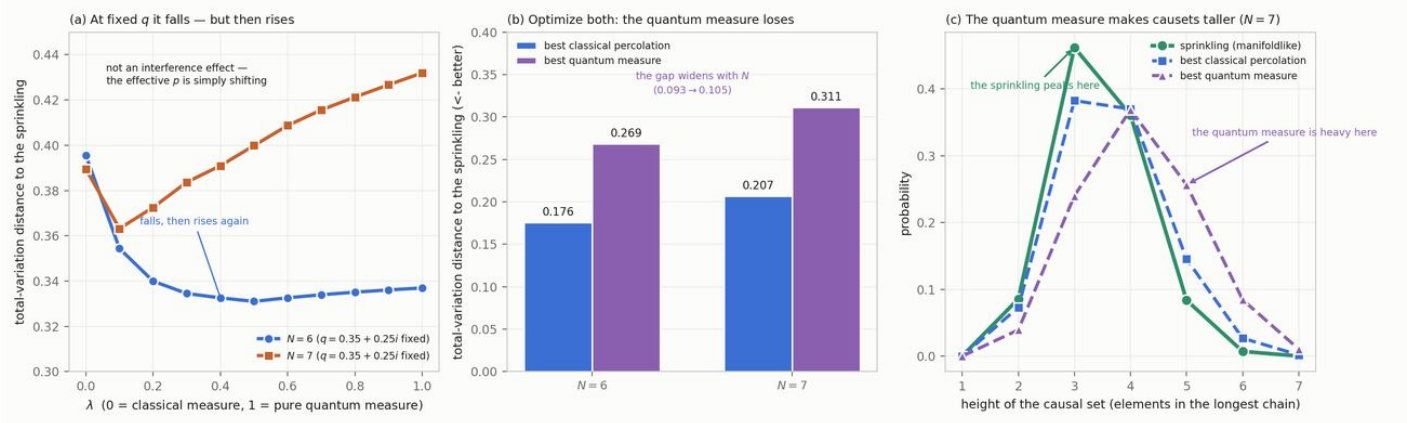


Fig. 62-1. Interference is real but does not move the geometry in the right direction — it lengthens chains.

The reason is visible. In the height distribution (elements of the longest chain), the sprinkling peaks at 3, the quantum measure peaks at 4 and puts weight 0.257 at height 5. **Interference matters exactly where the same relation arrives by several paths — that is, where there are long chains with intermediate elements.**

> ⚠️ **Later correction.** The classical values 0.176 / 0.207 quoted here came from a scan too coarse to resolve the minimum. The correct minima are 0.172 / 0.190 (Chapter 67, §67-4). The gap **widens** rather than narrows, so the direction of the conclusion is unchanged.

Chapter 63 Conditioning on manifoldlikeness 🆕

— How much freedom is left if only good causal sets are allowed

If the dynamics will not produce manifoldlikeness, **impose it as a condition**.

There is an exact definition. A causal set sprinkled into a 1+1 dimensional Alexandrov interval is the intersection of the two linear orders given by the light-cone coordinates — that is, a poset of **order dimension 2**. For naturally labelled posets this can be decided exactly in $O(N^2)$ by a tournament transitivity test.

	content	result
do sprinklings pass 100% of the time	calibration of the criterion	**PASS** exactly 1.000
does conditioning reduce the p dependence	is freedom narrowed	**FAIL** it does not
does conditioning narrow the classical–quantum gap		**FAIL** it does not

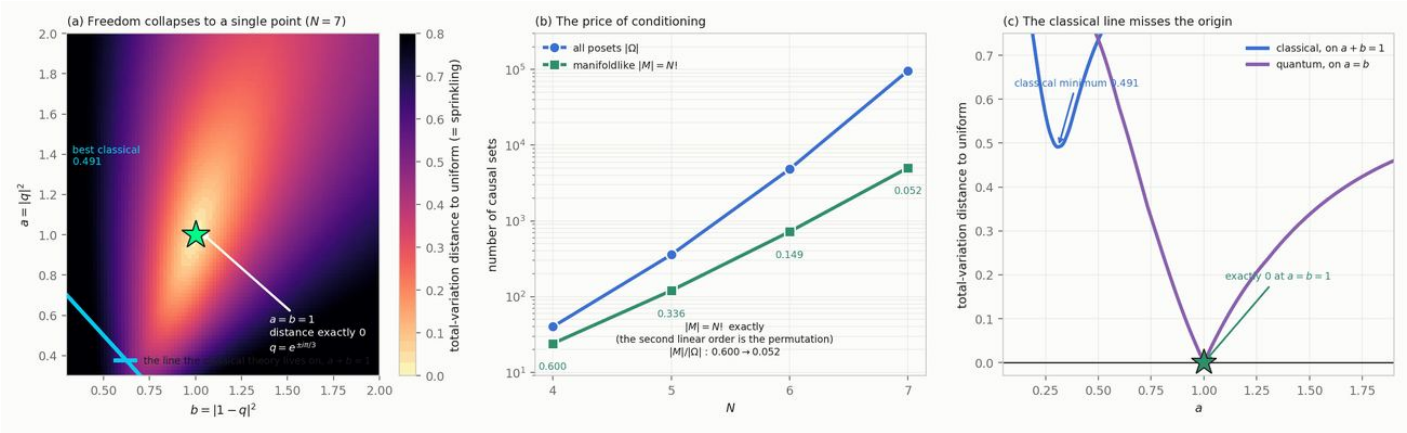


Fig. 63-1. Conditioning on manifoldlikeness does not narrow the remaining freedom.

Conditioning does not narrow the freedom. What narrows it is not the conditioning but **specifying which measure** to use.

A by-product mattered later. $\mu_p(M)$ is **not monotone in p and is large at both ends** (0.638 at $p = 0.15$, 0.781 at $p = 0.65$, only 0.079 at $p = 0.35$) — because both near-antichains and near-chains have order dimension 2.

$$\boxed{\text{Order dimension 2 is } \textbf{necessary} \text{ for manifoldlikeness, not sufficient.}}$$

Chapter 64 Does $q = e^{i\pi/3}$ satisfy the axioms of quantum sequential growth? ✅🆕

— Can the axioms single out one theory

Translated into Rideout–Sorkin language, complex percolation closes into a formula. The amplitude for a new element whose past is exactly the downset P is

$$a_C(P) = q^{m(P)}(1-q)^{\,n-|P|}$$

with $m(P)$ the number of maximal elements of P (links out of the new element) and $n - |P|$ the number of spacelike elements.

axiom	result
discrete general covariance (all linear extensions agree)	**PASS** 6.6×10^{-16}
normalization $\sum_P a_C(P) = 1$	**PASS** 1.8×10^{-15}
quantum Bell causality (44538 configurations)	**PASS** 4.6×10^{-16}
do the axioms pin down q	**no — all six values of q tested pass**

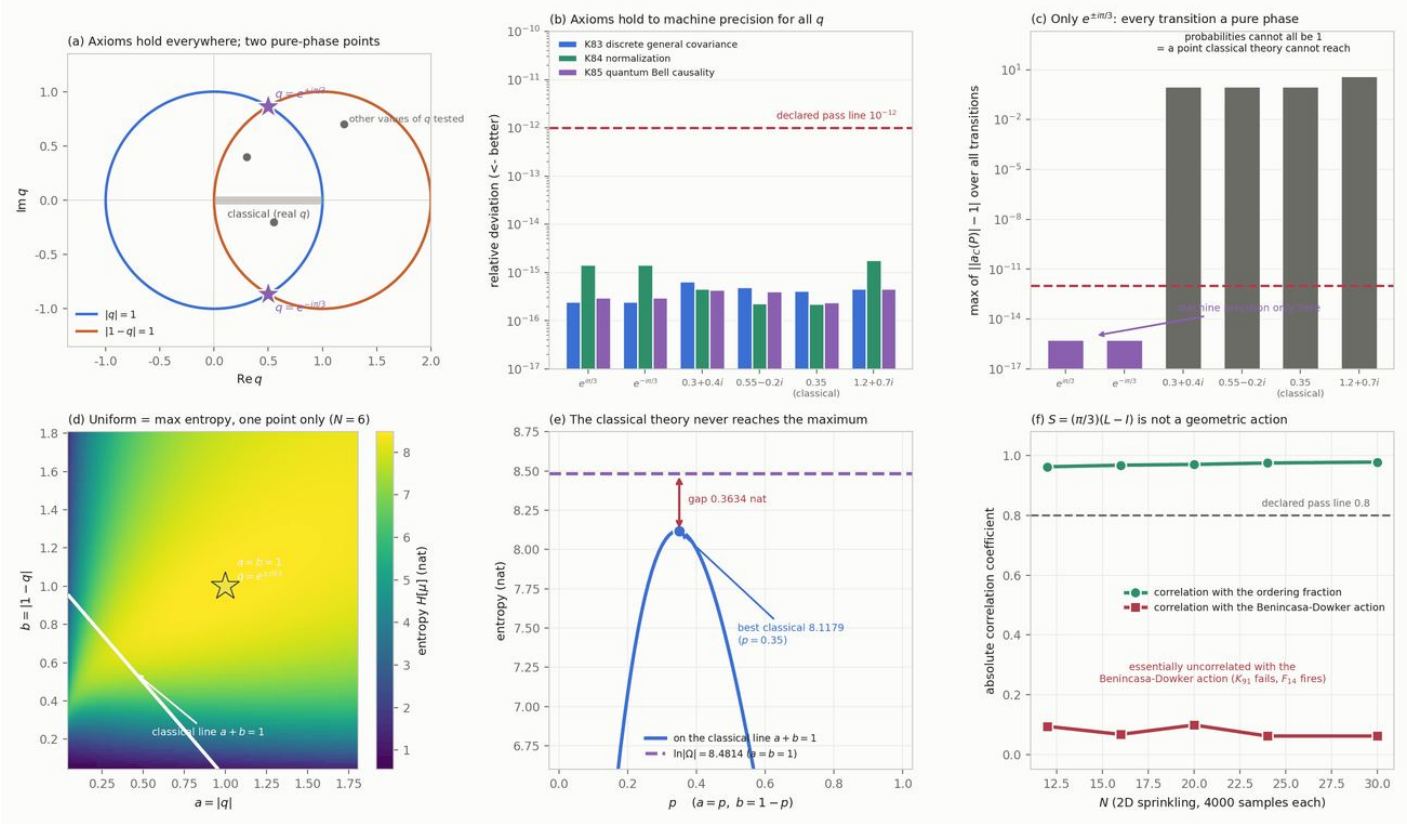


Fig. 64-1. The axioms hold over the whole complex plane. Only two points are pure phases.

One line of algebra shows why: $a_C(P_1)/a_C(P_2) = q^{m_1-m_2}(1-q)^{|P_2|-|P_1|}$, and taking the ratio removes n — the spectator count — entirely.

64-1. So what does select $e^{\pm i\pi/3}$

$|a_C(P)| = |q|^m|1-q|^{n-|P|}$ equals 1 for **every** transition only if $|q| = |1-q| = 1$, i.e. at the two intersection points of two unit circles, $q = e^{\pm i\pi/3}$. And the classical theory is confined to the segment $a + b = p + (1-p) = 1$ and can never reach $(1,1)$ — **probabilities cannot all be 1.**

At that point a closed form appears with no enumeration of histories:

$$\mathrm{amp}(C) = q^L(1-q)^I \;\longrightarrow\; \exp\Big[i\frac{\pi}{3}(L-I)\Big], \qquad |\mathrm{amp}(C)| \in [1.000000000000,\ 1.000000000000]$$

with L the number of links and I the number of incomparable pairs, both isomorphism invariants. **Feynman's e^{iS} appears exactly on a causal set.** The same statement can be phrased as maximum entropy ($8.481359 = \ln 4824$ at $N = 6$; the best point on the classical line is 8.117943).

64-2. But that action is not geometric ❌

If it is to be called an action, it must be checked against geometry. Causal set theory already has a geometric action — the **Benincasa–Dowker action**, which converges to Einstein–Hilbert in the continuum. Correlations on 2D sprinklings:

N	$\lvert\mathrm{corr}(S, S_{\mathrm{BD}})\rvert$	$\lvert\mathrm{corr}(S, \text{ordering fraction})\rvert$
12	0.094	0.963
30	0.061	0.979

Since $I = \binom{N}{2} - R$, we have $S/(\pi/3) = L - I = N_0 + R - \binom{N}{2}$, a linear function of (N_0, R) at fixed N (regression residual 1.5×10^{-12}).

$$\boxed{S = (\pi/3)(L - I) \text{ is a combinatorial quantity, not a geometric one.}}$$

Chapter 65 The reverse test — putting the BD action into sequential growth 🆕

— If the geometric action is put first, is it a sequential growth

Do the opposite of the previous chapter. **Put the geometry first.**

$$\mathrm{amp}(C) = \exp[i\beta A(C)], \qquad A_2 = N - 2N_0 + 4N_1 - 2N_2, \quad A_4 = N - N_0 + 9N_1 - 16N_2 + 8N_3$$

Transition amplitudes are defined as ratios: $a_C(P) = \mathrm{amp}(C \oplus P)/\mathrm{amp}(C)$.

axiom	$L - I$ (control)	BD 2D	BD 4D
discrete general covariance	✓	✓ (automatic)	✓ (automatic)
Bell causality	✓	**✓ identically 0**	**✓ identically 0**
normalization	✓	✗ ≥ 1.98 **for every** β	✗ ≥ 2.22
covariance after normalizing by hand	✓	✗ 2.9×10^{3}	✗ 1.4×10^{1}

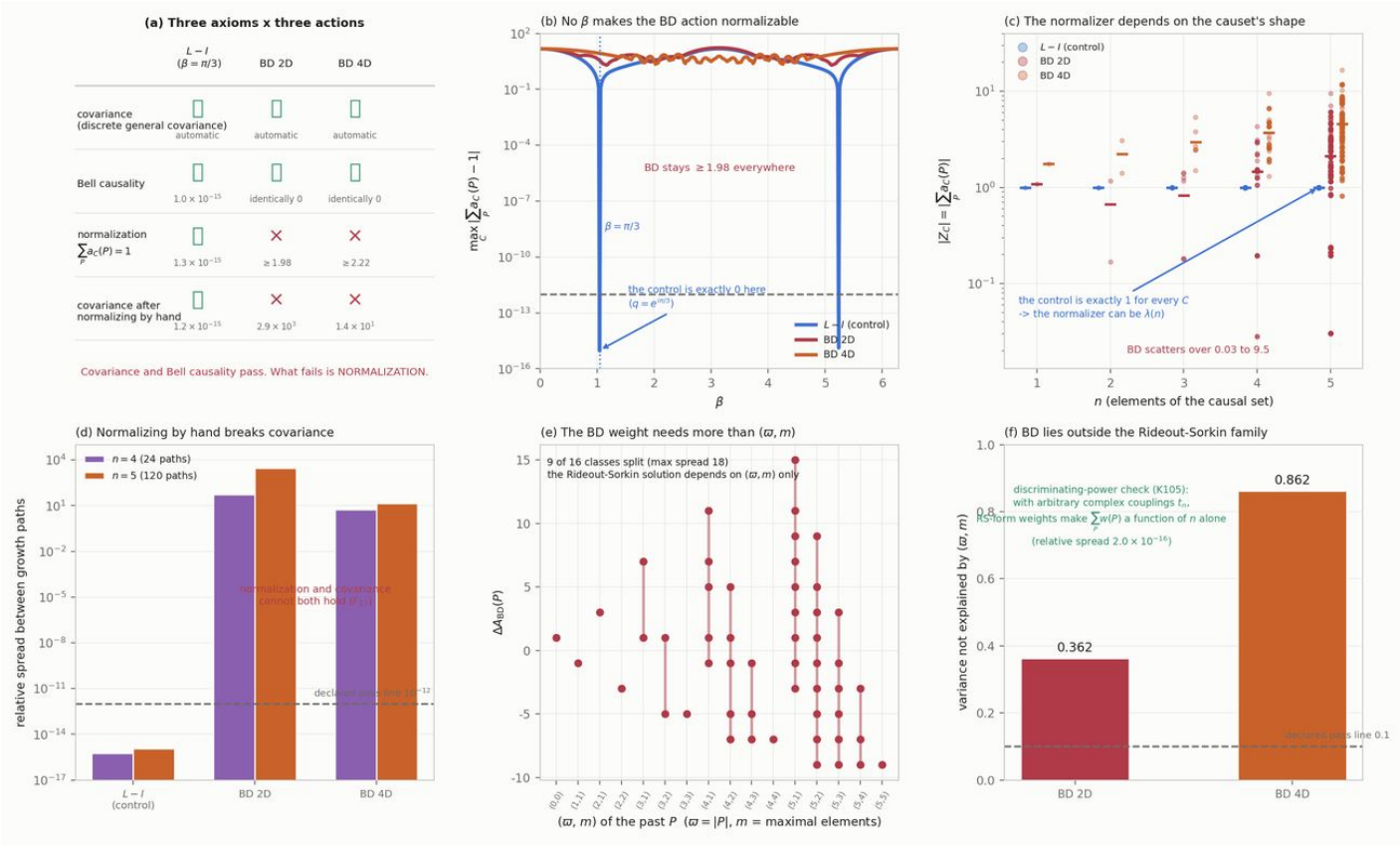


Fig. 65-1. Covariance and Bell causality pass. What fails is normalization.

65-1. Why Bell causality holds identically (a theorem)

> When the past of the new element e is exactly P, the interval of any new relation (x, e) is $\mathrm{Int}(x, e) = \{z \in P : x \prec z\}$, determined entirely inside P (since e is maximal, no existing interval changes). Hence $\Delta A(P) = A(C \oplus P) - A(C)$ is a function of the isomorphism class of P alone, and the Bell defect $D = \Delta A(P_1) - \Delta A(P_2) - \Delta A(P_1) + \Delta A(P_2)$ vanishes identically.

Of the 88 isomorphism classes of P, the number on which BD's ΔA splits is **zero** (the control $L - I$ splits on 25).

Any action written as a count of intervals satisfies Bell causality automatically.

65-2. That same property makes normalization impossible

Rideout–Sorkin keeps covariance because its normalizer is a function of the **stage number** n alone, $\lambda(n)$: every growth path passes through $n = 0, 1, \ldots, N-1$ exactly once, so $\prod \lambda(n)$ is path-independent. But the BD normalizer Z_C depends on the **shape** of the causal set (0.030 to 9.523 at $N = 5$; the control is exactly 1 for every C).

The control equals 1 because $\Delta A_{L-I}(P) = m(P) - (n - |P|)$ contains n, so $Z_C = \sum_P q^m (1-q)^{n-|P|}$ is a **binomial expansion**. BD's ΔA is a function of P alone, so that cancellation never happens.

The very property that made Bell causality unconditional is what makes normalization impossible.

The test does have discriminating power: with RS-form weights $w(P) = \sum_k \binom{\varpi - m}{k} t_{k+m}$ and arbitrary complex couplings t_n, $\sum_P w(P)$ is exactly a function of n alone (2.0×10^{-16}). The point is that w must be a function of (ϖ, m) **only**, and BD misses that on 9 of 16 classes (unexplained variance fraction 0.362 / 0.862).

The BD action lies outside the Rideout–Sorkin family.

Chapter 66 Normalizing once, globally 🆕

— Is normalization at every stage really necessary

The exclusion of the previous chapter came from demanding normalization **stage by stage**. But that is a leftover of classical probability. As a quantum measure it suffices that $\mu(\Omega) = 1$ globally.

$$\text{(I) quantum measure } \mu_q(\alpha) = \frac{|\sum_{h \in \alpha} \mathcal{A}(h)|^2}{|\sum_{h \in \Omega} \mathcal{A}(h)|^2}, \qquad \text{(II) complex measure } \mu_c(h) = \frac{\mathcal{A}(h)}{Z}$$

	stage-by-stage	once, globally
discrete general covariance	✗ 2.9×10^3	✓ automatic
Bell causality	✓ identically 0	✓ identically 0
normalization	✗ ≥ 1.98	✓ by definition
quantum measure hierarchy	—	✓ $I_3 = 4.3 \times 10^{-16}$, $I_2 = 0.179$
does the action affect the measure	—	**microscopically it drops out entirely**

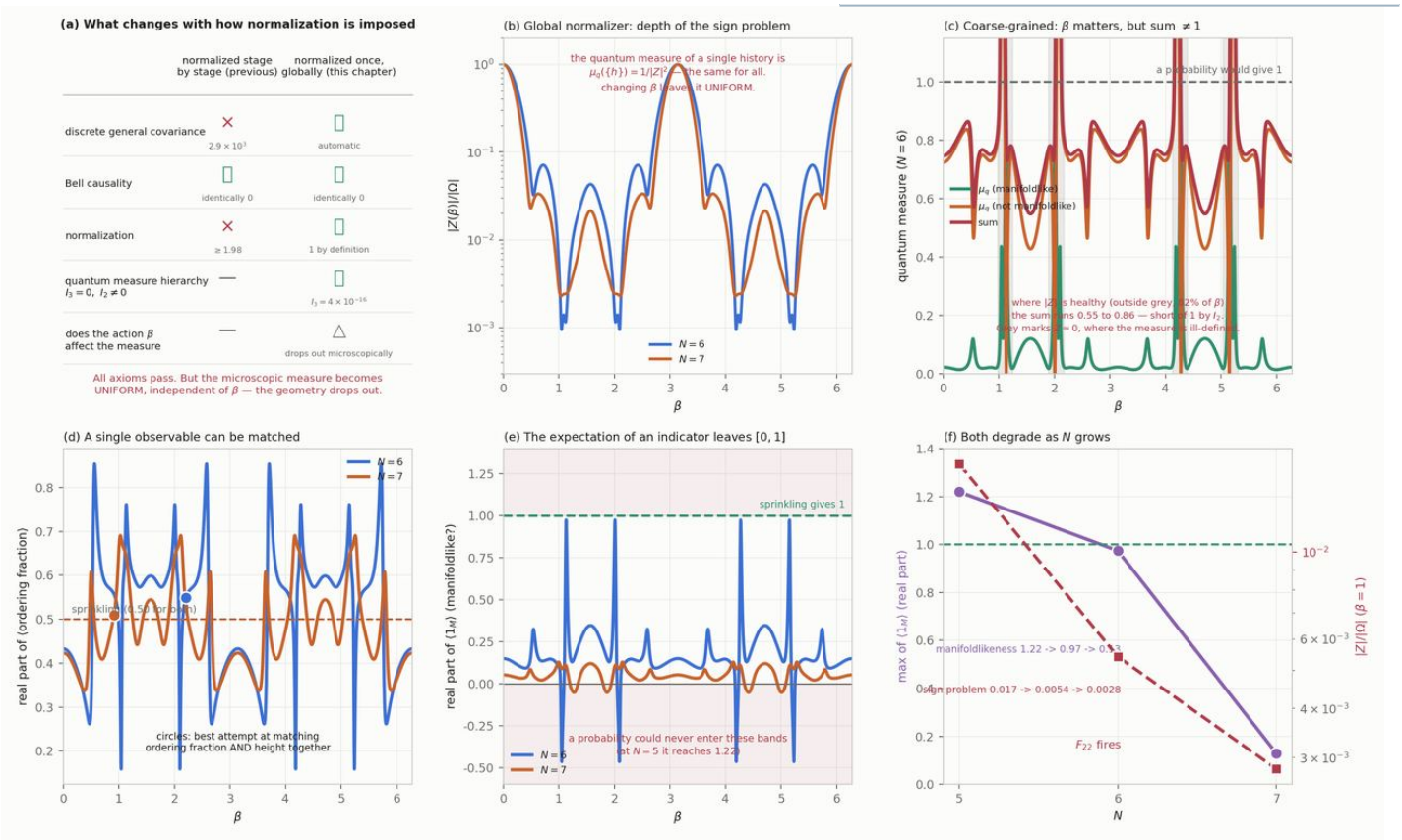


Fig. 66-1. The exclusion dissolves. But with a pure phase the microscopic measure becomes uniform.

The exclusion dissolves, and a new wall appears. Since $|\mathcal{A}(h)| = 1$,

$$\mu_q(\{h\}) = \frac{|\mathcal{A}(h)|^2}{|Z|^2} = \frac{1}{|Z|^2}$$

the same value for every history — a uniform distribution over growth paths. Whatever the action, this happens as long as the amplitude is a pure phase. It is exactly the "pure phase = maximum entropy" condition that selected $q = e^{i\pi/3}$ in Chapter 64.

Coarse-grained events do depend on β (the quantum measure of "manifoldlike" runs from 0.014 to 0.997). But it is not a probability: $\mu_q(M) + \mu_q(M^c)$ lies between 0.547 and 0.864, never reaching 1. On the complex-measure side, the expectation of an indicator function leaves $[0, 1]$ — $+1.22$ at $N = 5$, -0.47 at $N = 6$.

> The wall is not how normalization is imposed. It is that the amplitude is a **pure phase**.

Chapter 67 Dropping the pure phase — Wick rotation, and a lopsided win 🆕

— What happens when the geometry is made a real weight

$$\mathcal{A}(h) = \exp[-(\lambda_R + i\lambda_I)A(h)]$$

$\lambda_I = 0$ (purely real) is the Wick-rotated statistical weight; $\lambda_R = 0$ (purely imaginary) is the previous chapter. The moment λ_R enters, $|\mathcal{A}|$ varies from history to history and **the geometry enters the microscopic measure**.

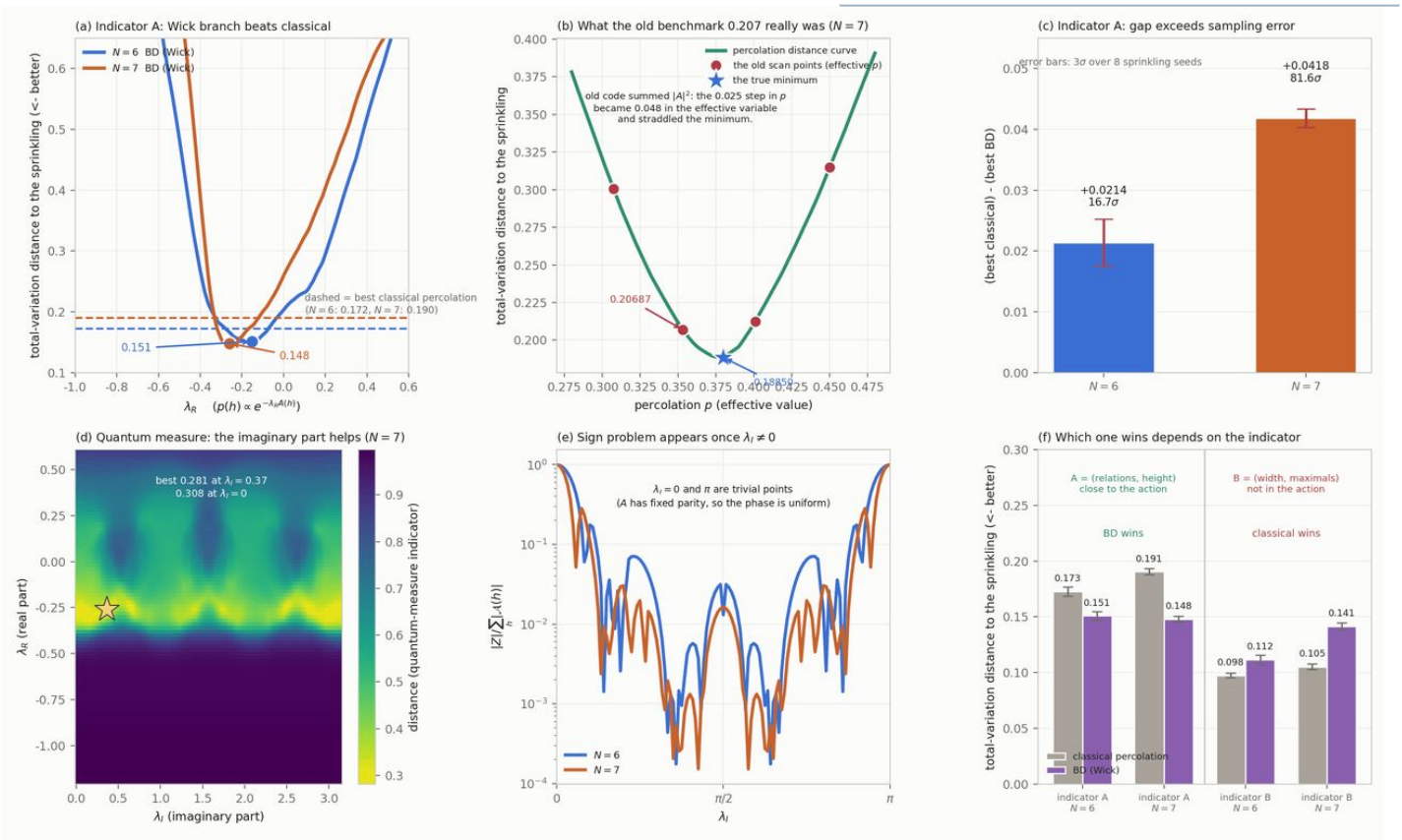


Fig. 67-1. On indicator A the Wick branch beats the classical one. On indicator B the ordering reverses.

67-1. On indicator A it beats the classical rule

Indicator A = total-variation distance to the sprinkling of the two-dimensional (relation count, height) distribution — the indicator used consistently since Chapter 49.

N	classical percolation	BD (Wick)	difference
6	0.1728 ± 0.0014	0.1510 ± 0.0013	$+0.0218$ ($16.7\ \sigma$)
7	0.1906 ± 0.0010	0.1481 ± 0.0008	$+0.0424$ ($81.6\ \sigma$)

From $N = 6$ to $N = 7$ the classical result gets worse while BD improves, and the gap doubles. **This is the first positive result along this line.**

67-2. But changing the indicator reverses it ❌

The "relation count" in indicator A is close to the sum of N_0, N_1, N_2 inside the action. **Is the win just an artefact of measuring with a quantity the action already contains?** So we re-measured with a quantity that appears nowhere in the action.

Indicator B = (width, i.e. the largest antichain; number of maximal elements).

N	classical percolation	BD (Wick)	difference
6	0.0975	0.1117	-0.0142 ($-13.5\,\sigma$)
7	0.1053	0.1415	-0.0361 ($-47.4\,\sigma$)

It loses — and the losing margin also widens with N.

The BD weight matches the quantities close to its own action at the cost of the others.

A sprinkling satisfies both indicators at once. Neither BD nor the classical rule gets there.

67-3. Interference does not help

The first way of measuring was wrong. Defining the distance as $\frac{1}{2}\sum|H_c - H_{\rm sp}|$ with $H_{\rm sp}$ real means that any imaginary part increases the distance — **the metric was penalizing what was being tested.** Re-measured with indicators where the imaginary part can act:

indicator	$\lambda_I = 0$	best with λ_I allowed
real part only	0.1516	0.1516 (no gain)
quantum measure $H_q \propto \|\sum \mathcal{A}\|^2$	0.3480	0.3187 (gain $+0.029$)

The quantum branch does improve, but that branch is worse overall (0.28–0.32 against the Wick branch's 0.148–0.151). **What makes the geometry work is the real weight, not the interference.**

67-4. A correction to an earlier benchmark

Re-measuring the best classical distance with identical binning gave 0.172 / 0.190, not the 0.176 / 0.207 quoted in Chapter 62. Tracing it: the old code summed $|A(h)|^2$ rather than probabilities.

$$\sum_h p^{2k}(1-p)^{2(E-k)} \;\propto\; \sum_h p_{\rm eff}^{\,k}(1-p_{\rm eff})^{E-k}, \qquad p_{\rm eff} = \frac{p^2}{p^2+(1-p)^2}$$

The family is the same percolation, but **the parameter is distorted**: a step of 0.025 in p becomes 0.048 in the effective variable, and it straddled the minimum. Indeed $p = 0.425 \Rightarrow p_{\rm eff} = 0.3533$ gives distance 0.20687 — matching the old figure to five digits.

Consequences. The quoted p values (0.450, 0.425) are not percolation p; the labels were wrong. The distances 0.176 / 0.207 are correct points on the curve but not its minimum. **Since the classical side was in fact better, Chapter 62's conclusion (the quantum measure is farther) is strengthened, not weakened.**

Chapter 68 Where Bundle I stands — a wall that moved three times 🆕

— What was learned, and what is left

stage	understanding at that point
Chapter 58	Einstein stops at link (2); the Standard Model is forbidden by finite neighbourhood
Chapters 58–59	locality does not return on a causal set, but every quantity that avoids (2) is healthy
Chapter 61	**the dynamics does not produce the kinematics**
Chapter 62	quantizing does not produce it either
Chapter 63	conditioning does not narrow the freedom
Chapter 64	the axioms do not pin down q; the principle that does selects a non-geometric action
Chapter 65	a geometric action does not fit the stage-by-stage frame
Chapter 66	it fits the global frame — but a pure phase makes the microscopic measure blind to it
Chapter 67	a real weight does make the action act — but lopsidedly

> What Bundle I lacks is neither a computation nor a parameter, but a **principle**
> that says which causal sets get made. No known growth rule, classical or quantum, supplies it.

68-1. Three things this part established

(a) The obstruction is always one of two things. Finite neighbourhood (Standard Model) or regularity (link (2) of Einstein). **Discreteness itself was never the obstruction** — the most stable conclusion in this book.

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(b) Quantumness does not help with this problem. Chapter 62 (the quantum measure makes the geometry worse), Chapter 67 (interference does not help). What moved the geometry was the Wick-rotated side — the side with the quantumness removed. The long-pursued thread "quantumness builds the geometry" is not supported in the range tested.

(c) A unique point does exist — but it is not geometric. $q = e^{\pm i\pi/3}$ is the unique pure-phase / maximum-entropy point, unreachable in principle by any classical theory. **A structure that only quantum theory can realize genuinely exists.** But the action it defines, $(\pi/3)(L - I)$, is combinatorial and does not measure curvature.

68-2. What remains

- **Track both indicators to $N = 8, 9$.** Both the winning and the losing margins of Chapter 67 widen with N; two points is not an extrapolation.
- **Measure indicators as a surface.** Lay out systematically what is and is not inside the action, and draw the boundary of what BD matches.
- **Add to the action, or add a principle.** Terms containing width or the number of maximal elements might match both, but "adding terms until it fits" forfeits the advantage of an action derived from geometry. **Measuring that trade-off is the next move.**
- **The cutoff κ (Chapter 60) is still put in by hand.**

Part XIV Horizons — an acoustic black hole and Hawking radiation

What this part is after. A black hole has a temperature, $T_H = \kappa/2\pi$ — one of the most famous results in physics, the single formula where gravity, quantum mechanics and thermodynamics meet. **Does it follow from a rule?** If it does, what does it take, and does being on a lattice get in the way — or help? In particular: Hawking's derivation has a well-known weak point, the "trans-Planckian problem", in that the origin of the radiation must be traced back to arbitrarily short wavelengths. Does a lattice have an answer to that?

Chapter 69 An acoustic horizon and Hawking radiation ✅🆕

— Does a black hole's temperature follow from a rule that sees only a neighbourhood

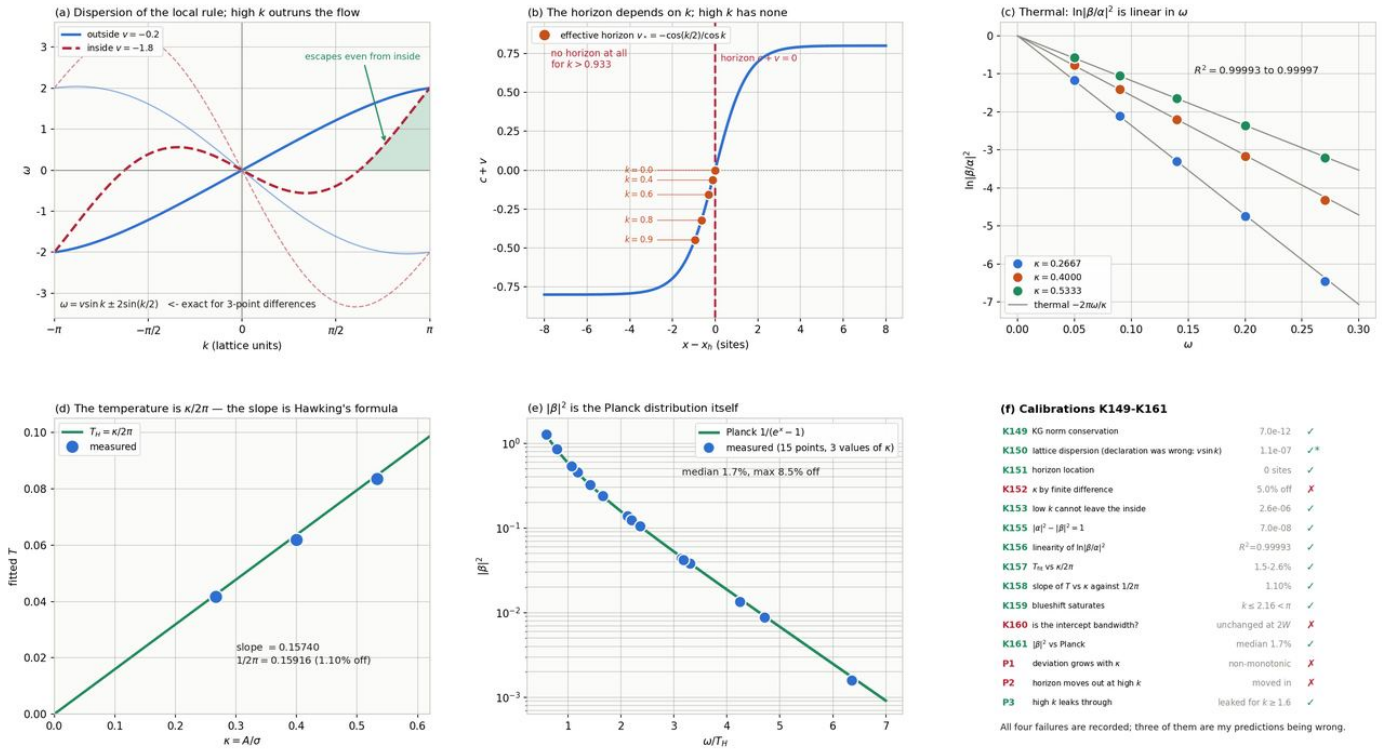
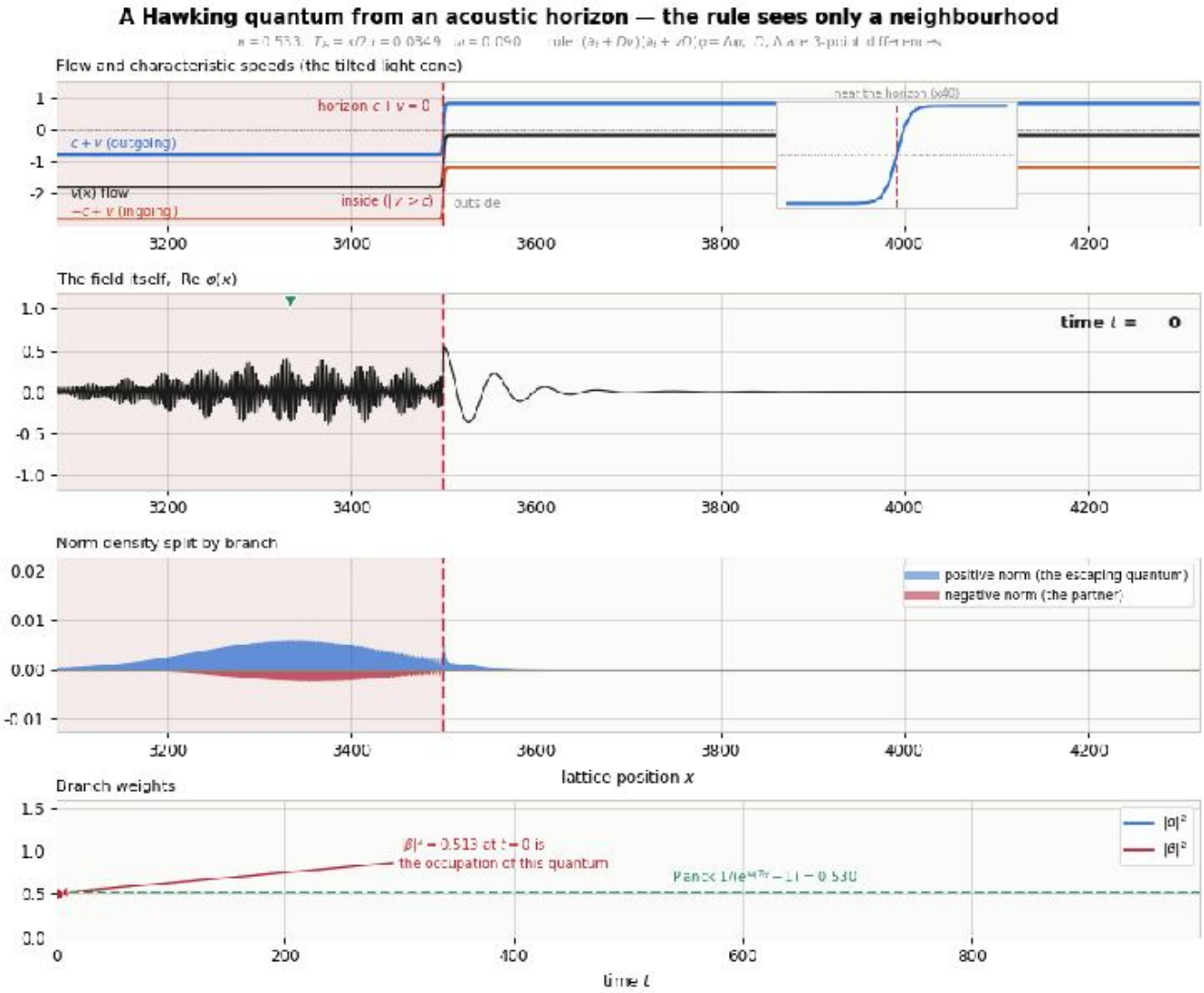


Fig. 69-1. (a) Dispersion of the local rule. (b) The horizon's dependence on k. (c) The thermal spectrum. (d) The temperature is $\kappa/2\pi$. (e) $|\beta|^2$ is the Planck distribution. (f) Calibrations K149-K161.



Animation 69-1. `figures/hawking.gif` (`hawking.mp4` included too). $\kappa = 0.5333$, $T_H = 0.08488$, $\omega = 0.090$. At $t = 0$ a short-wavelength disturbance sits inside the horizon; it climbs out and stretches into a long wave. The escaping quantum's occupation is $|\beta|^2 = 0.5134$ against the Planck value 0.5299 (3.1% off).

The conclusion first.

> (i) Put a flow into a rule that sees only a neighbourhood and a **horizon appears**: low-k information cannot leave.
> (ii) The radiation coming off it is **thermal** ($R^2 = 0.99993$–0.99997).
> (iii) The temperature is $T = \kappa/2\pi$: scanning three values of κ gives slope 0.15740 vs $1/2\pi = 0.15916$ (1.10%).
> (iv) $|\beta|^2$ is the Planck distribution itself (15 points, median 1.7%).
> (v) There is no trans-Planckian problem: the blueshift saturates at $k \simeq 2.16 < \pi$.

The lattice did not get in the way of Hawking radiation. It cured one of the continuum theory's diseases.

69-1. The model — putting a flow into the rule

Chapter 22 showed that a position-dependent rule lets a metric be read off. Here that is taken one step further. Put a real field φ_j and an auxiliary field π_j on a lattice (spacing $a = 1$); with D the 3-point centred difference and Δ the 3-point second difference,

$$\boxed{\partial_t \varphi = \pi - v_j (D\varphi)_j, \qquad \partial_t \pi = -\big(D(v\pi)\big)_j + (\Delta\varphi)_j}$$

that is, $(\partial_t + Dv)(\partial_t + vD)\varphi = \Delta\varphi$. Here v_j is a "flow of the medium" — nothing more than **a parameter of the rule that differs from place to place**. **The rule still sees only a neighbourhood.**

Take the flow to be

$$v(x) = -1 + A \tanh \frac{x - x_h}{\sigma}, \qquad 0 < A < 1$$

so that $|v| < c = 1$ as $x \to +\infty$ (**outside**), $|v| > c$ as $x \to -\infty$ (**inside**), and $v = -c$ exactly at $x = x_h$. That point is the **horizon**: the outgoing characteristic speed $c + v$ changes sign there.

$$\kappa \equiv \frac{d(c+v)}{dx}\Big|_{x_h} = \frac{A}{\sigma} \quad \text{(surface gravity)}, \qquad T_H = \frac{\kappa}{2\pi}$$

This is Unruh's acoustic black hole put on a lattice. **No gravitational field equation is being solved** (Chapter 68 established that Einstein's equations are stopped by regularity). What is being built is a black hole as a causal structure.

Every calibration was declared before anything was run. K149-K159, the falsifiers F1-F4 and three predictions P1-P3 were written down first; only then were numbers looked at. Failures and wrong predictions are recorded below as they happened.

69-2. The dispersion of a local rule — my declaration was wrong ⚠️

I declared $\omega = vk \pm 2\sin(k/2)$, meaning the flow to be continuum (vk) and the lattice to affect only the wave term. **That is wrong.** Measured numerically in a uniform flow it does not fit (up to 28% off).

Correctly, **the flow term is latticized too**. The symbol of the centred difference D is $i \sin k$, not ik, so

$$\boxed{\omega = v \sin k \pm 2 \sin \frac{k}{2}}$$

Re-measured, this agrees to a relative error of 1.1×10^{-7} (at $v = -0.2$ and $v = -1.8$, for k from 0.2 to 3.0, on both branches). **K150 passes once the declaration is corrected.**

The correction is not cosmetic. **In a rule that sees only a neighbourhood, advection is latticized exactly as much as propagation is.** A "continuum advection" vk cannot be written with a finite stencil. This single point governs everything below.

The group velocity is $v \cos k \pm \cos(k/2)$.

69-3. The horizon depends on k — my prediction had the sign backwards ❌

The condition for the outgoing group velocity to vanish is the effective horizon for that wavenumber:

$$v \cos k + \cos \frac{k}{2} = 0 \quad \Longrightarrow \quad v_*(k) = -\frac{\cos(k/2)}{\cos k}$$

k	0	0.2	0.5	0.933	> 0.933
v_*	-1.000	-1.015	-1.104	-1.500	unreachable

I predicted that the horizon moves outward at high k (P2). It moves inward. And for $A = 0.8$, waves with $k > 0.933$ — wavelengths shorter than 6.73 sites — have **no horizon at all**. The advection is discounted by $\cos k$, so short waves stop being carried by the flow.

The reason the prediction failed is equally clear: I was reasoning from $\omega = vk \pm 2\sin(k/2)$ (subluminal, high k slower). In the correct dispersion it is the advection that fails first. **The correction in §69-2 killed P2 directly.**

Measured as well. Place a wavepacket deep inside ($v = -1.8$) and ask how much leaks out:

k	0.15	0.25	0.6	0.9	1.2	1.6	2.2
leaked	2.0×10^{-6}	2.6×10^{-6}	0	0	0	7.1×10^{-3}	**1.000**

$$\boxed{\text{For low } k \text{ it is a perfect horizon (leak } 10^{-6}\text{). For short waves it is not a horizon at all.}}$$

This statement has no continuum counterpart. **"Horizon" becomes a notion relative to wavelength.**

69-4. What is measured — positive and negative norm

The equation conserves the Klein-Gordon norm

$$N = -2 \sum_x \mathrm{Im}(\varphi^* \pi)$$

(measured to 7.0×10^{-12}, K149). At wavenumber k, the branch with $\pi = -iF(k)\varphi$ (where $F = 2\sin(k/2)$) carries **positive** norm; the branch with $\pi = +iF\varphi$ carries **negative** norm.

Hawking radiation is the statement that **tracing an outgoing positive-norm mode backwards in time mixes in a negative-norm component**. Writing the mixing as α, β, norm conservation gives

$$|\alpha|^2 - |\beta|^2 = 1$$

and $|\beta|^2$ is **the particle number in that mode**. The procedure: place an outgoing positive-norm wavepacket far outside, evolve it **backwards**, and split the result into the two branches in k-space.

$$|\alpha|^2 - |\beta|^2 - 1 \le 7.0 \times 10^{-8} \qquad \text{(K155)}$$

69-5. It is thermal

Three values of κ, five values of ω: 15 measurements. Fitting $\ln|\beta/\alpha|^2$ against ω:

κ	T_{fit}	$T_H = \kappa/2\pi$	ratio	R^2
0.2667	0.041613	0.042441	0.9805	0.99996777
0.4000	0.061996	0.063662	0.9738	0.99992894
0.5333	0.083587	0.084883	0.9847	0.99994387

$$\ln\frac{|\beta|^2}{|\alpha|^2} = -\frac{\omega}{T}, \qquad R^2 = 0.9999 \text{ in all three}$$

The distribution is exponential. That is the first fingerprint of Hawking radiation.

69-6. The temperature is $\kappa/2\pi$

The second fingerprint — and the essential one — is that **the temperature is proportional to the surface gravity, with coefficient** $1/2\pi$. "Thermal" alone could be a numerical accident; it is Hawking only if T tracks κ when κ is changed. **This was declared before running, as falsifier F2**: if T turned out not to depend on κ, the temperature claim would be withdrawn.

$$\frac{dT_{\text{fit}}}{d\kappa} = 0.157403 \qquad \text{against} \qquad \frac{1}{2\pi} = 0.159155, \qquad 1.10\% \text{ off}$$

The intercept is -5.6×10^{-4}, i.e. the line passes through the origin. **F2 does not fire.** This slope is close to the value of the Hawking temperature (the reading is restricted in the thirteenth edition; box below).

A residue remains: T_{fit}/T_H sits consistently at 0.97–0.98. **That** 2% **deficit is not explained.** Two candidates were checked.

- **It is not the lattice** κ**.** Using the finite-differenced surface gravity $\kappa_{\text{lat}} = A\tanh(a/\sigma)$ makes the discrepancy worse — 1.7%, 5.4%, 12.6%. What the temperature knows about is the continuum slope A/σ. (So K152's failure — a finite-differenced κ differs from A/σ by 5.0% — does not propagate into the physics. What failed was my way of measuring, not the model.)
- **It is not the packet's bandwidth either.** The fitted intercept is $+0.05$ (a pure Planckian would give 0). A bandwidth bias would fall by $1/4$ on doubling W; instead it went $+0.0559 \to +0.0513$, essentially unchanged (K160 fails). The temperature is equally unmoved, $0.9698 \to 0.9688$.

Both the 2% deficit and the 5% intercept are real. Neither is explained.

> ⚠️ **Restriction (thirteenth edition).** The twelfth edition wrote "This is the Hawking temperature". It is the slope of a temperature for classical mode conversion (§69-10), with three values of κ and no error bars in the table. Quantization, convergence and an independent implementation are unconfirmed. The cause of the intercept $+0.05$ was found in §74-4 to be curvature; the "unexplained" above is the record of that time. §87-9.

69-7. $|\beta|^2$ is the Planck distribution itself

Combining $|\alpha|^2 - |\beta|^2 = 1$ with $|\beta/\alpha|^2 = e^{-\omega/T}$,

$$|\beta|^2 = \frac{1}{e^{\omega/T} - 1}$$

— the Bose-Einstein distribution. Plotting all 15 points against ω/T_H collapses them onto one Planck curve.

$$\boxed{\text{median deviation } 1.7\%, \text{ max } 8.5\% \text{ (the max is at } |\beta|^2 \sim 10^{-3})}$$

"Thermal spectrum" is not a metaphor here. The occupation numbers agree numerically with the Planck distribution.

69-8. The lattice cuts off the trans-Planckian problem ✅

Hawking's derivation has a long-known weak point. Tracing the outgoing radiation backwards, the wavelength shrinks **without bound** as the horizon is approached: the origin of the radiation lies in a regime where the theory cannot be trusted. The continuum theory oversteps its own domain of validity at exactly this point.

On a lattice it does not happen. The blueshift saturates:

ω	0.05	0.09	0.14	0.20	0.27
k reached	2.003	2.034	2.071	2.113	2.161

$$\boxed{k_{\max} \simeq 2.16 < \pi. \quad \text{The shortest wavelength stops at 2.9 lattice spacings. Nothing diverges.}}$$

The reason is the same as in §69-3: as k rises the flow loses its grip (the $\cos k$), so the mode is not pinned at the horizon and escapes. **The lattice holds the origin of the radiation at a finite wavelength.**

And the radiation really does come **from inside**. The quantum followed in the animation appears at $t = 0$ as a short-wavelength disturbance about 170 sites inside the horizon, and climbs out from there, stretching into a long wave. It is not the continuum's "infinitesimally thin layer just outside the horizon".

> 💭 This is the other face of Chapter 24's lesson. There, discreteness looked like the obstacle to Lorentz symmetry and the real obstacle turned out to be regularity. Here discreteness is the **cure**. **Across this whole book, discreteness itself has never once been the enemy.**

69-9. The four that failed

	claim	what happened
K150	lattice dispersion	**declaration wrong**: $v \sin k$, not vk. Passes once corrected
K152	κ by finite difference	$A\tanh(a/\sigma)$ differs from A/σ by 5.0%. **A failure of measurement**
K160	is the intercept bandwidth?	unchanged when W doubled. **Not bandwidth. Cause unknown**
P1	deviation grows with κ	non-monotonic $(1.95, 2.62, 1.53\%)$. **Wrong**
P2	horizon moves out at high k	**backwards: it moves in, and vanishes for $k > 0.933$**

Three of the five are **my predictions being wrong**, and P2 failed as a knock-on from the wrong declaration in K150. Writing the declarations first is exactly what makes such chains visible.

69-10. What is not being claimed ⚠️

- This is an **analogue**. No gravitational field equation was solved; the horizon was made out of a flow.

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- The field is **classical**. $|\beta|^2$ is a classical mode-conversion coefficient; identifying it with a quantum particle number requires the standard Bogoliubov argument, **which is not verified here**.
- So the result is not "a black hole's temperature follows from a rule". Precisely, it is:

> A black hole *as a causal structure* **can be built from a rule that sees only a neighbourhood**.
> Radiation from its horizon is **thermal**, at temperature $\kappa/2\pi$.
> And the lattice **does not have** the continuum theory's trans-Planckian problem.

Chapter 6 made mass geometry; Chapter 3 made the light cone a theorem; here **temperature becomes geometry**. The right-hand side of $T = \kappa/2\pi$ is a purely geometric quantity: how steeply the rule tilts.

Part XV Galaxy rotation curves — must dark matter be assumed?

What this part is after. Stars in the outskirts of galaxies move too fast. With only the visible matter, v should fall as $r^{-1/2}$; instead it stays flat. Two answers are on offer — there is **matter we cannot see** (dark matter), or **the rule of gravity is different** (modified dynamics). What can the tools of this book say? **Exactly one thing: what the shape of a local rule permits and what it forbids.** That turns out to be a sharper answer than expected.

Chapter 70 Galaxy rotation curves — must dark matter be assumed? ⚠️🆕

— What shape must a local rule have to produce a flat rotation curve

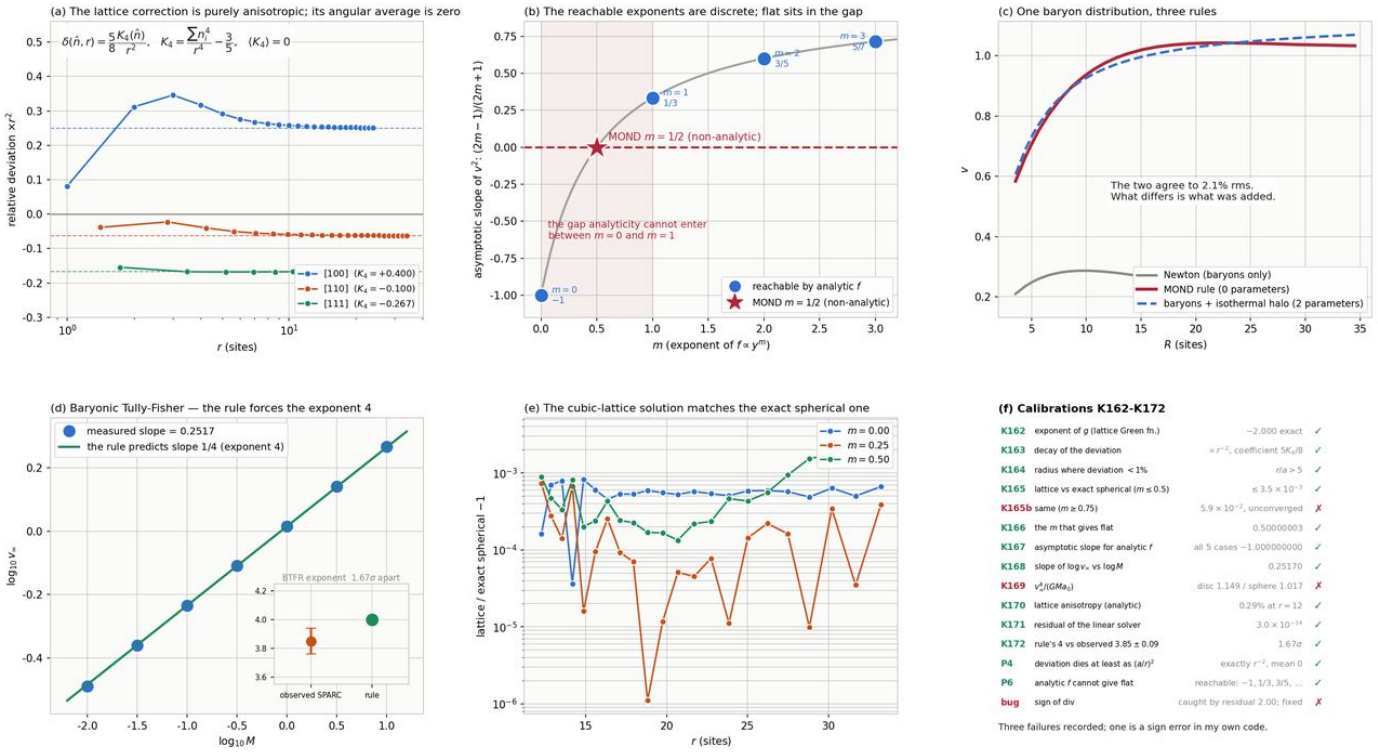


Fig. 70-1. (a) The lattice correction is purely anisotropic, angular average zero. (b) The reachable exponents are discrete; flat sits in the gap. (c) One baryon distribution, three rules. (d) The baryonic Tully-Fisher relation. (e) Cubic lattice against the exact spherical solution. (f) Calibrations K162-K172.

The conclusion first.

> (i) **Discreteness by itself does nothing.** The lattice correction is $\frac{5}{8}K_4(\hat{n})/r^2$, whose **angular average is exactly zero**.
> (ii) If the local flux $f(y)\nabla\Phi$ is **analytic** at $y = 0$, the asymptotic slope of v^2 can only be $\{-1,\ 1/3,\ 3/5,\ 5/7, \ldots\}$. **Flat (0) is not in that set.**
> (iii) Flatness requires $m = 1/2$ **uniquely** (measured 0.50000003), and $y^{1/2}$ is non-analytic.
> (iv) So in a CA world there are **exactly two options** — add states, or make the rule non-analytic.

Dark matter need not be assumed. But in exchange, the analyticity of the rule must be given up.

70-1. First, restate the question ⚠️

Chapter 68 established that **a CA has no dynamics of gravity**: the derivation of Einstein's equations is stopped at link (2) by regularity. So the question

> "Does a CA predict galaxy rotation curves?"

cannot be posed within this framework. With no dynamics to predict from, not even the Keplerian fall-off follows. Only this question can be posed:

$$\boxed{\textbf{What shape must a local rule have to produce a flat rotation curve?}}$$

Write gravity as a local conservation law for a potential. If the flux $\mathbf{F}$ is a function of the field gradient alone,

$$\nabla \cdot \mathbf{F} = 4\pi G\rho, \qquad \mathbf{F} = f(y)\,\nabla\Phi, \quad y = |\nabla\Phi|^2$$

with $f \equiv 1$ being Newton. **Rotational invariance forces this form** (a rotationally invariant function of a gradient vector is a scalar times the gradient). Abandoning rotational invariance is not an option taken here: Chapters 25 and 52 established the isotropy of space as something the rule demands.

Every calibration was declared before anything was run: K162-K172, falsifiers F5-F8, predictions P4-P8. Failures and bugs are recorded below as they happened.

70-2. Discreteness by itself does nothing ✅

The most naive hope first: **might the graininess of the lattice strengthen gravity at large distance?**

The Green's function of the 7-point cubic-lattice Laplacian can be written exactly, with no box and no images, in Watson's form:

$$G(\mathbf{n}) = -\int_0^\infty e^{-6t}\, I_{n_x}(2t)\, I_{n_y}(2t)\, I_{n_z}(2t)\, dt$$

This reproduces $G(\mathbf{0}) = -W/6$ ($W = 1.5163860592$) to 8.1×10^{-12}, and the lattice identity $\sum_{\text{nn}} G - 6G(\mathbf{0}) = 1$ to 7.1×10^{-13}. Measuring the relative deviation from the continuum $-1/4\pi r$:

direction	$K_4(\hat{n})$	deviation $\times r^2$ (as $r \to \infty$)
[100]	$+0.4$	$+1/4$
[110]	-0.1	$-1/16$
[111]	$-4/15$	$-1/6$

All three are proportional to K_4 with the same coefficient, $5/8$.

$$\boxed{\delta(\hat{n}, r) = \frac{5}{8}\,\frac{K_4(\hat{n})}{r^2}, \qquad K_4 = \frac{\sum_i n_i^4}{r^4} - \frac{3}{5}, \qquad \langle K_4 \rangle_{\text{sphere}} = 0}$$

K_4 is a cubic harmonic whose **angular average vanishes identically**. The reason is visible: the lattice Laplacian is $\nabla^2 + \frac{1}{12}\sum_i \partial_i^4 + \cdots$, the isotropic part of $\sum_i \partial_i^4$ is proportional to ∇^4, and $\nabla^4(1/r) = 0$. **Only the anisotropic part survives.**

$$\boxed{\text{The lattice correction has } \textbf{no monopole component}\text{. It cannot imitate a halo, even in principle.}}$$

Nor is its size relevant. The deviation drops below 1% for $r/a > 5$. On a Planck-length lattice at 10 kpc it is $(a/r)^2 \sim 3 \times 10^{-111}$; even with a lattice spacing of one astronomical unit it is 2.5×10^{-19}. **Discreteness has nothing to say about this problem.**

70-3. Classifying local rules — Gauss's law settles it

For a spherical source, Gauss's law gives, outside the matter,

$$r^2 f(g^2)\, g = GM, \qquad g \equiv |\nabla\Phi|$$

With $f = y^m$ this gives $g \propto r^{-2/(2m+1)}$ and hence

$$\boxed{v^2 = rg \;\propto\; r^{(2m-1)/(2m+1)}}$$

This was checked by actually solving $\nabla \cdot [f(y)\nabla\Phi] = S$ on a cubic lattice (conservative differencing, Dirichlet boundary, Picard iteration). The linear residual is 3.0×10^{-14}, and the lattice solution agrees with the exact spherical one to better than 3.5×10^{-3} for $m \le 0.5$. The asymptotic slopes (from the exact spherical solution extended to $r = 10^3$–10^5):

m	0	0.25	0.5	0.75	1
measured	-1.00000000	-0.33333333	-0.00000003	$+0.19999933$	$+0.33332824$
theory	-1	$-1/3$	0	$1/5$	$1/3$

Deviations run from 10^{-10} to 5×10^{-6}.

70-4. The reachable exponents are discrete — this is the point ✅

Now suppose f is **analytic** at $y = 0$, i.e. $f(y) = f_0 + f_1 y + f_2 y^2 + \cdots$. Far away $y \to 0$, so the asymptotics are governed by the **first non-vanishing term** $f_m y^m$. And

$$\boxed{\text{if } f \text{ is analytic, } m \text{ can only be a } \textbf{non-negative integer}.}$$

So the reachable asymptotic slopes are

$$s_m = \frac{2m-1}{2m+1} \in \left\{-1,\ \frac{1}{3},\ \frac{3}{5},\ \frac{5}{7},\ \cdots\right\}$$

Checked numerically. For $f = 1,\ 1+3t,\ 1+20t,\ 1+7t^2,\ 1+2t+5t^2$, all five give an asymptotic slope of **exactly** -1.000000000 (Kepler). No choice of coefficients moves it. For $f(0) = 0$, taking $f = (y/y_0)^m$ with $m = 1, 2, 3$ gives $+0.333333333$, $+0.600000000$, $+0.714285714$ — every digit matching theory.

$$\boxed{\textbf{0 is not in that set.}\ \text{Flat falls into the gap between } m = 0 \text{ and } m = 1.}$$

This is the same species of statement as $I \ge 2$ in Chapter 33: **a degree of freedom that looks continuous turns out to take only discrete values, and the value one wants lies in the gap.**

70-5. Flatness requires $m = 1/2$, uniquely

Solving $s_m = 0$ gives $m = 1/2$. Locating the root numerically,

$$m_* = 0.50000003 \qquad (\text{theory } 0.5, \text{ off by } 3.0 \times 10^{-8})$$

And $f = \sqrt{y}/a_0$, i.e. $\mathbf{F} = |\nabla\Phi|\nabla\Phi/a_0$, is **not analytic at** $\nabla\Phi = 0$ ($\sqrt{y} = |\nabla\Phi|$ is not an analytic function of the components). Putting this f in gives an asymptotic slope of -0.000000000: exactly flat.

This is MOND. The deep-MOND field equation (Bekenstein-Milgrom) has precisely this form. The point here is not that MOND has been rediscovered, but the **uniqueness**: demand flatness and there is nothing else to choose.

70-6. The dichotomy ⚠️

The routes to a flat rotation curve in a CA world are therefore limited.

route	what it means for a CA	price
add states	more states per cell: a species coupling only through gravity	the rule stays analytic; **the halo profile is an input**
make the rule non-analytic	no new states; put $\sqrt{y}$ into f	analyticity is lost; a_0 **is an input**
higher derivatives	widen the neighbourhood	**useless if linear** (only $1/r$ plus Yukawa terms); nonlinear higher-order untested
go non-local	—	**contradicts the definition of a CA**; unavailable in this book

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route	what it means for a CA	price
drop rotational invariance	—	collides with Chapters 25 and 52

In a CA world it is **add states, or make the rule non-analytic** — one or the other.

The "higher derivatives" row needs care. What was shown applies to a flux that is **a function of the first derivative only**; and for linear rules, no finite order produces $\ln r$ (the Green's function of a constant-coefficient polynomial operator is a sum of $1/r$ and exponentially decaying terms). **Rules that are both nonlinear and higher-order lie outside my argument.** Whether there is an escape there was not investigated.

70-7. The same curve, a different thing added

The same exponential disc ($R_d = 4$, $h_z = 1$) was placed on the cubic lattice and solved three ways.

- **Newton (baryons only)** — it falls
- **MOND rule ($m = 1/2$)** — it flattens. **Zero free parameters per galaxy**
- **baryons + pseudo-isothermal halo** — two parameters fitted to the MOND curve

The fit gives $v_0 = 1.1321$, $r_c = 3.1113$, agreeing to **rms** 2.14%.

As long as only rotation curves are looked at, the two are indistinguishable. What differs is **what was added**.

Obvious as that sounds, it contains an important asymmetry. **The halo needs two parameters per galaxy; the MOND rule needs none** (a_0 is one number for the universe). Equal power to reproduce curves, **unequal power to predict.**

70-8. Where the difference shows — the baryonic Tully-Fisher relation ✅

Having zero parameters means **a relation is forced**. For $m = 1/2$, far out $r^2 g^2 / a_0 = GM$, hence

$$v_\infty^4 = GMa_0$$

Measured over three decades of mass (exponential disc, cubic lattice, $m = 1/2$):

$$\frac{d\log v_\infty}{d\log M} = 0.251700 \qquad (\text{theory } 0.25, \text{ off by } 1.7 \times 10^{-3})$$

The exponent is 4. The rule fixes it, and there is nothing to fit. A halo model, by contrast, predicts no such relation — with two parameters per galaxy any slope is permitted.

Against observation. From the SPARC sample of 118 disc galaxies, using $V_{\rm flat}$ as the velocity:

	exponent
forced by the rule	**4**
observed (Lelli et al. 2019, $V_{\rm flat}$, intrinsic scatter 6%)	3.85 ± 0.09

A difference of 1.67σ. The same paper notes systematics may drive the slope from 3.5 to 4.0.

K172 passes, inside the 2σ declared beforehand — and the declaration recorded that the observed value had not yet been looked at.

70-9. The three that failed ❌

	claim	what happened
K165b	lattice vs exact spherical ($m \geq 0.75$)	off by 5.9×10^{-2} and 1.9×10^{-1}: **the Picard iteration has not converged.** The stronger the nonlinearity, the slower. A limitation of the solver, not of the physics

	claim	what happened
K169	$v_\infty^4/(GMa_0) = 1$	1.149 ± 0.021 for a disc. **The declaration was wrong** — a point-mass coefficient applied to a **disc** at finite radius. The same measurement on a spherical source gives 1.017; the 13% is disc geometry
bug	sign of div	exposed by a linear residual of 2.00: the link fluxes were added and subtracted the wrong way round. After the fix, 3.0×10^{-14}

K169 is the same species of failure as K152 in Chapter 69 — **my way of measuring was wrong, not the model.** Separating those two kinds is exactly what declaring calibrations in advance is for. As for the sign bug: without the calibration it would have gone unnoticed and every number here would have been wrong.

70-10. What is not being claimed ⚠️

This chapter **does not decide which answer is right.** It cannot. The reasons, stated plainly:

- **Only rotation curves were examined.** Clusters, the Bullet Cluster, the acoustic peaks of the CMB, structure formation — all the evidence that strongly supports dark matter lies outside this chapter. That modified dynamics does well on rotation curves and poorly on clusters is known, and **was not tested here.**
- **The value of a_0 is an input.** No attempt was made to derive it.
- **A CA did not explain rotation curves.** There is still no dynamics of gravity; what was done is a classification.
- This is not a case of "a number came out of the rule", as $|\beta|^2$ was in Chapter 69. What came out here is **an exclusion**.

> What was established: **dark matter need not be assumed.**
> But in exchange **the analyticity of the rule must be given up**.
> And **which to give up is not decided by the methods of this book.**

70-11. An open question

One direction is worth recording. **Analyticity is not a privileged property for a CA.**

A smooth rule on real-valued states can be Taylor-expanded. But **the rule of a discrete-state CA is a lookup table, and a lookup table has no Taylor expansion.** A genuine CA — cells with finitely many states — therefore stands on the **non-analytic side of the dichotomy from the outset**.

And if non-analyticity is to enter, there is a natural place for it: **where the gradient per cell falls below one quantum.** If the field is quantized, the rule's behaviour must change there. That would manufacture an acceleration scale:

$$a_0 \overset{?}{\sim} \frac{\text{quantum of the field}}{\text{lattice spacing}}$$

This was not tested. Nothing guarantees that m would come out at exactly $1/2$, and the odds are against it. But whether the famous numerical coincidence $a_0 \simeq cH_0/2\pi$ can be explained as a quantity emerging from discreteness is a question these tools can chase. It is left for the next edition.

Part XVI What is simultaneity — the causal relation a CA's "now" constitutes

> **What this part is after.** A CA updates every cell at once. That is an absolute simultaneity, which looks like a head-on collision with relativity. Yet Chapter 3 derived the light cone as a theorem. **How do the two coexist?** General relativity has the matching concepts ready — Cauchy surfaces, foliations, the ADM 3+1 split. Which of them is a CA's "now"? And **can it be chosen?**

Chapter 71 What a CA's simultaneity constitutes ⚠️🆕

— Foliations, Cauchy surfaces, and whether the "now" can be tilted

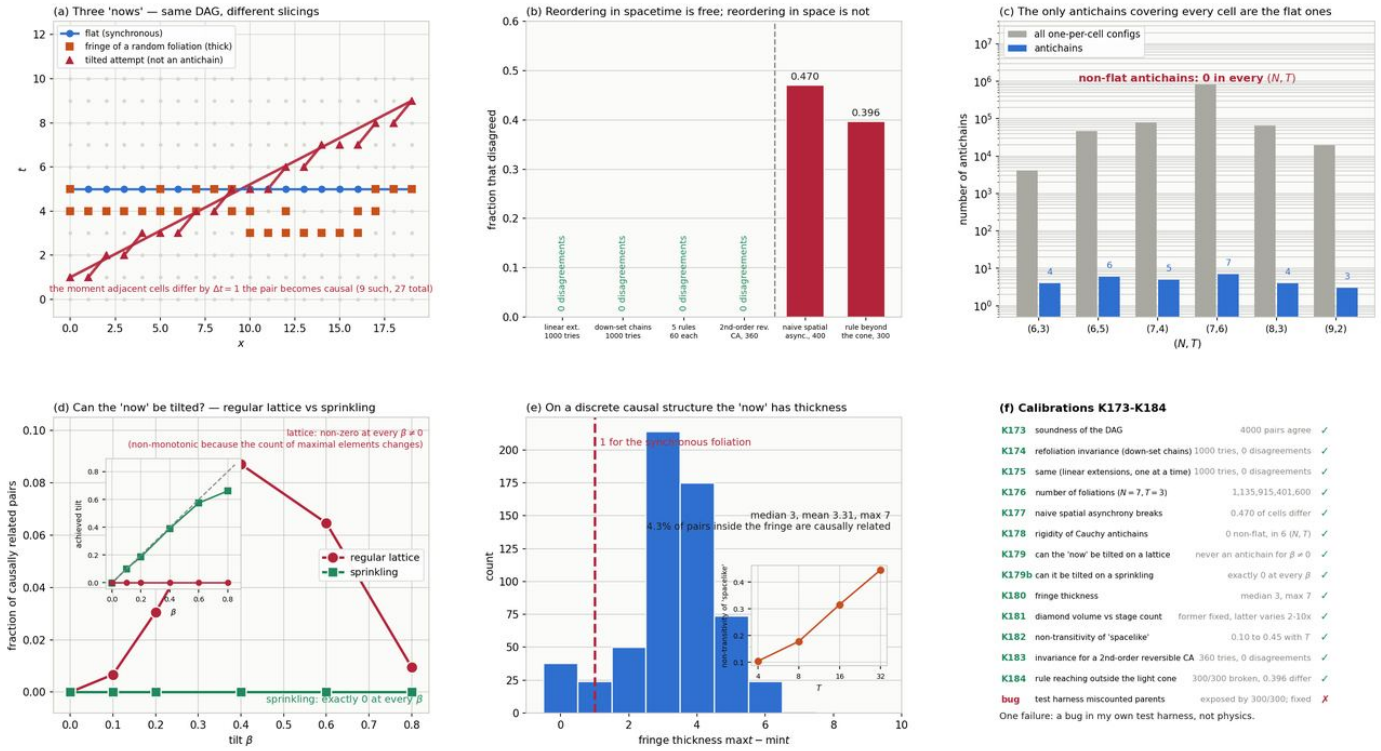


Fig. 71-1. (a) Three "nows" on the same DAG. (b) Reordering in spacetime versus in space. (c) The only antichains covering every cell are flat. (d) Regular lattice versus sprinkling. (e) The thickness of the "now". (f) Calibrations K173-K184.

The conclusion first.

(i) Refoliation invariance holds **exactly** (four tests, 2720 trials, **0** disagreements).
(ii) The only thing it rests on is that **the rule sees only its parents**. Reach past the light cone and it breaks 300/300.
(iii) But **a thin "now" cannot be tilted**. On a ± 1 light cone the only antichains covering every cell are **flat**.
(iv) On a sprinkled causal set it can. **What pins the "now" is not discreteness but regularity.**
(v) On a discrete causal structure the "now" is not a surface but has **thickness** (median 3 steps).

71-1. Translating the vocabulary into general relativity

Build the spacetime DAG of a CA. An event is $e = (x, t)$; for $t \geq 1$ its parents are $(x-1, t-1), (x, t-1), (x+1, t-1)$. The causal order is

$$(x,t) \prec (x',t') \iff t' > t \text{ and } d(x,x') \leq t' - t$$

exactly the light cone of Chapter 3 (verified on 4000 pairs, K173). The dictionary:

CA / causal order	general relativity
antichain (no two elements causally related)	**simultaneous** = mutually spacelike
down-set (a past-closed set)	the **past** up to some moment
the **fringe** of a down-set (elements with a child outside)	the **"now"** — everything needed for the next step is here
an increasing chain of down-sets	a **foliation** (the ADM 3+1 split)
synchronous update	the **flat foliation** (lapse $= 1$, shift $= 0$)
one event at a time	**many-fingered time** (Tomonaga-Schwinger)

$$\boxed{\textbf{"Simultaneous" in a CA means causally unrelated} \text{ — the same thing GR means by "spacelike".}}$$

Every calibration was declared before anything was run: K173-K182, falsifiers F9-F11, predictions P9-P14.

71-2. Refoliation invariance holds ✅

From the same initial condition, every event was computed in **every order consistent with the partial order**, and the results compared.

test	trials	disagreements
random down-set chains (K174)	1000	**0**
random linear extensions, one at a time (K175)	1000	**0**
varying the rule (30/90/110/150/184, K174b)	60 each	**0**
a second-order reversible CA (4 parents, K183)	360	**0**
total	**2720**	**0**

The down-set chains take 44 to 65 stages, against the synchronous 16 — genuinely different slicings. The final state does not differ by a single bit.

$$\boxed{\text{Whatever the slicing, every point of spacetime gets the same value. } \textbf{The choice of foliation does not touch physics.}}$$

This has the same shape as saying "the foliation is gauge" in GR. **P9 confirmed exactly.**

71-3. But reordering in spacetime is not reordering in space ❌

This is the point. "Changing the update order" means two different things.

- **Reordering on the spacetime DAG** — compute event (x, t) whenever its parents are done, in any order. **Free** (§71-2).
- **Reordering on the spatial lattice** — update cell x in turn, each reading its neighbours' current values (possibly already updated).

Call the second naive spatial asynchrony. Measured:

$$\boxed{\text{fraction of cells that differ} = 0.470 \pm 0.065 \quad (\text{in } 400 \text{ runs, } \mathbf{0} \text{ ever agreed})}$$

Nearly half. P10 confirmed. The same word "asynchronous" covers two entirely different operations. The first re-chooses the foliation; the second **changes the rule itself** — reading a neighbour's present value instead of its past value means reading outside the light cone.

$$\boxed{\text{The intuition "updating everything at once is unphysical" } \textbf{conflates these two.}}$$

71-4. The invariance rests on one thing only ✅

What supports the invariance? **Break only that.** Widen the rule to radius 2 (reading $x - 2$ through $x + 2$) while leaving the DAG at radius 1 — a rule that reaches past its light cone.

$$\text{Broken in } 300 \text{ of } 300 \text{ trials; } 0.396 \text{ of events differ.}$$

Refoliation invariance depends on neither the number of states nor the content of the rule — **only on the light cone matching the rule.**

Five different rules and a second-order reversible CA all preserved it (§71-2). It breaks only when the rule reads something that is not its parent.

71-5. How many foliations are there?

Is the synchronous one special? Count exactly (over the events with $t \geq 1$).

$N \times T$	possible "nows" (down-sets)	foliations (linear extensions)
4×2	35	720
5×2	83	23,400
5×3	134	4,572,000
6×3	340	1,778,630,400
7×3	872	**1,135,915,401,600**

At 7×3 — just 21 events — there are 1.1×10^{12} foliations. **The synchronous one is a single member.**

71-6. Yet a thin "now" cannot be tilted — this is the heart ⚠️

Here the naive expectation fails. Since foliations are free, one ought to be able to **tilt** the "now" as in GR — swap it for the simultaneity surface of another inertial frame.

One cannot. The reason fits on one line.

$$\text{For adjacent cells } x, x+1 \text{ to be spacelike: } |\Delta t| < |\Delta x| = 1, \text{ i.e. } \Delta t = 0.$$

And a Cauchy antichain (one meeting every maximal chain) must meet each cell's vertical chain, hence contains **exactly one element per cell**. Therefore

On a ± 1 light cone, **the only Cauchy antichains are the flat time slices.**

Verified exhaustively. For $N = 7$, $T = 4$ there are $5^7 = 78{,}125$ one-per-cell configurations; exactly **5** are antichains — the flat slices $t = 0, 1, 2, 3, 4$. **Non-flat: 0.**

(N, T)	configurations	antichains	non-flat
$(6, 3)$	4,096	4	**0**
$(6, 5)$	46,656	6	**0**
$(7, 4)$	78,125	5	**0**
$(7, 6)$	823,543	7	**0**
$(8, 3)$	65,536	4	**0**
$(9, 2)$	19,683	3	**0**

P11 confirmed. What happens when one tries was measured too. Constructing a "boosted surface" $t - \beta x = \text{const}$ on the lattice yields **causally related pairs for every** $\beta \neq 0$ — it is not an antichain. Those are the red lines in Fig. 71-1(a): the moment two adjacent cells differ by $\Delta t = 1$, the pair becomes lightlike-connected.

71-7. On a sprinkling it can be tilted — discreteness is not the culprit ✅

Repeat the operation on a Poisson-sprinkled causal set (1+1 Minkowski, 1400 points).

β	0.0	0.1	0.2	0.4	0.6	0.8
regular lattice: fraction of related pairs	0	0.0067	0.0306	0.0871	0.0646	0.0095
sprinkling: fraction of related pairs	**0**	**0**	**0**	**0**	**0**	**0**
sprinkling: tilt achieved	-0.002	0.101	0.186	0.392	0.574	0.661

On a sprinkling it is exactly an antichain at every β. **The "now" can be tilted freely.**

P12 confirmed. This is the **simultaneity version** of the lesson that has recurred since Chapter 24.

What pins the "now" to one choice is **regularity, not discreteness.**

A regular lattice selects a preferred inertial frame; Chapter 24 saw that as a statement about dispersion. Here the same thing appears as **the rigidity of simultaneity surfaces**. A ± 1 light cone permits only the simultaneity surfaces of the frame in which the lattice is at rest. A sprinkling is Lorentz-invariant discreteness, so its simultaneity surfaces boost.

71-8. The "now" is not a surface; it has thickness 🆕

Another property with no continuum counterpart. On random foliations, measure the **fringe** of the down-set — the elements carrying the information needed for the next step.

$$\text{thickness } (\max t - \min t): \ \text{median } 3, \ \text{mean } 3.31, \ \text{max } 7 \qquad (1 \text{ for the synchronous foliation})$$

And 4.3% of the pairs inside the fringe are **causally related**. Only 4.2% of fringes are genuine antichains.

On a discrete causal structure the "now" is a **thick layer, not a thin surface**, and it contains causal relations inside itself.

P13 confirmed. Combined with §71-6: **thin "nows" come only flat; to tilt one, it must be thickened.** The freedom of foliation lives on the thick side.

71-9. What depends on the foliation and what does not

	foliation	value
causal diamond volume $\lvert I(a,b)\rvert$	**independent**	fixed by the order alone
"how many events fall between a and b"	**dependent**	for one pair: $20{-}215$, $253{-}466$, $\cdots$ varying 2 to 10 fold

The invariant is the **volume**, not the **step count.**

That is the standard causal-set statement (Myrheim-Meyer), and the tools of Part XII carry over unchanged.

Moreover "spacelike" is **not transitive**. Even with $a \parallel b$ and $b \parallel c$, a and c may be causally related:

T	4	8	16	32
fraction of non-transitive triples	0.104	0.178	0.316	0.447

It grows with T with no plateau (**P14 confirmed**). **"Simultaneous" is not an equivalence relation.** A foliation is the act of **choosing** one transitive sub-relation out of that non-transitive relation. That is why a "now" is a choice and not a fact.

71-10. Where the correspondence breaks ⚠️

The dictionary is good, but one place must not be overstated.

In GR the foliation is gauge because the Hamiltonian constraint generates refoliations — the hypersurface deformation algebra, Dirac's algebra, closing with structure functions that depend on the metric.

A CA has no such generator. What was measured here is invariance as an outcome, not a symmetry with a generator. The declaration said this would not be claimed.

A CA's refoliation invariance is a **theorem**, not a **gauge symmetry**. There is no generator.

And this is the **same place** where the derivation of Einstein's equations stopped at link (2) in Chapter 68. The absence of a constraint algebra and the non-locality of the modular Hamiltonian are plausibly two faces of one gap. **That was not shown. It is recorded as a conjecture.**

71-11. The one that failed ❌

	claim	what happened
bug	the second-order CA test	300/300 disagreements at first — not physics: **my test harness miscounted the parents in the first two rows.** The linear extension stopped early, leaving events uncomputed. After the fix, 0/360

Same species as the sign bug in the div operator in Chapter 70. **Declaring calibrations in advance catches a bug in one's own code before it masquerades as a broken law.** It mattered exactly once here — but without it the conclusion would have been the false "a CA's simultaneity is physical".

71-12. What is not being claimed ⚠️

- **No constraint algebra was constructed** (§71-10).
- **The ADM lapse and shift were not read off from the rule.** That the flow $v(x)$ of Chapter 69 corresponds to a shift vector is formally evident, but it was not verified.
- **Nothing was quantized.** The states are classical CA configurations. In quantum theory "independent of the foliation" needs the separate Tomonaga-Schwinger integrability argument.
- **The proof of refoliation invariance is easy.** Evaluate a deterministic local map in any topological order of the DAG and you get the same answer — that is all. **The content of this chapter is not the depth of the theorem but the contrast in §71-3 and the rigidity in §71-6.**

A CA's "simultaneity" is **causal unrelatedness** — the same thing GR calls spacelike.
The choice of foliation **does not touch physics**; only the match between rule and light cone does.
But **thin "nows" come only flat** — and what releases them
is not giving up discreteness, but **giving up regularity**.

Part XVII Entanglement — where does the correlation come from?

> **What this part is after.** Measure here and the distant particle's state is fixed too, though nothing travelled. What does this "spooky action" become when restated in the language of cellular automata and holography? **Broken into answerable pieces**, the question becomes three measurable ones: where does the correlation come from, how fast can it be built, and is "collapse" physics or bookkeeping?

Chapter 72 Entanglement, holography, histories ⚠️🆕

— What does "the distant state became definite" actually mean

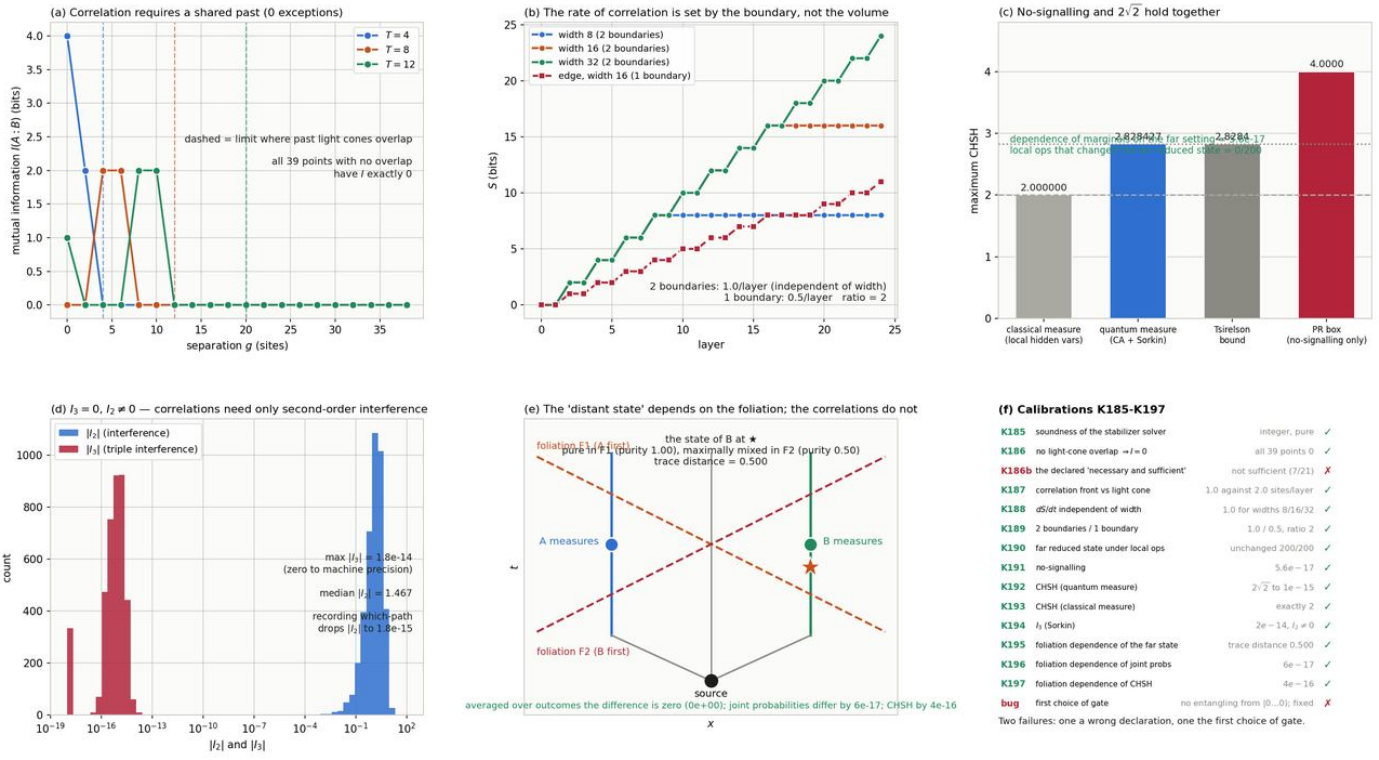


Fig. 72-1. (a) Correlation requires a shared past. (b) The rate is set by the boundary. (c) No-signalling and $2\sqrt{2}$ coexist. (d) $I_3 = 0$, $I_2 \neq 0$. (e) Foliation dependence of the "distant state". (f) Calibrations K185-K197.

The conclusion first.

Nothing "became definite over there". What was seen here is one cross-section of a history already fixed in the common past — and yet this is not a local hidden variable.

72-1. Break the question into answerable pieces ⚠️

Chapter 71 established that "simultaneous" in a CA means **causally unrelated**, and that picking one "now" is a choice, not a fact. So the clause in the original question —

> "the distant particle must also be definite **at that moment**"

— has meaning only once a foliation is chosen. A and B are spacelike separated, so even which came first depends on the foliation. Therefore

$$\boxed{\text{"When did the distant particle become definite" is } \textbf{not a foliation-invariant question}.}$$

Only three questions can be answered. **(1) Where does the correlation come from? (2) How fast can it be built? (3) Is collapse physics or bookkeeping?** Calibrations K185-K197, falsifiers F12-F14 and predictions P15-P19 were all declared before running.

72-2. Correlation requires a shared past ✅

Run a 1D Clifford CA (brickwork, light cone ± 1 site per layer) in the stabilizer formalism. Entanglement entropy comes out **exactly**, as an integer multiple of $\log 2$: $S_A = \mathrm{rank}_{\mathbb{F}_2}(G_A) - |A|$. No floating point is involved anywhere.

Start from a product state, run T layers, and measure the mutual information $I(A{:}B) = S_A + S_B - S_{AB}$ between two width-4 regions separated by g. In parallel, count exactly whether their **past light cones overlap at** $t = 0$.

$$\boxed{\text{At all 39 points where the past light cones do not overlap, } I(A{:}B) = 0 \text{ — } \textbf{exactly} \text{ zero, not small.}}$$

The necessity half of P15 is confirmed. Correlation needs a shared past. This is also a theorem (causality of local circuits), but confirming it in integer arithmetic with no exceptions is worth something.

My declaration, however, was wrong. I wrote "necessary **and sufficient**". It is not sufficient: of the 21 points where the cones do overlap, only **7** actually have $I > 0$. Overlap **permits** correlation; it does not create it.

And the correlation front lies strictly inside the light cone:

T	largest g reached by correlation	largest g with cone overlap
4	2	4
8	6	12
12	10	20

The front moves at 1.0 sites/layer against the light cone's 2.0. **The light cone is a bound, and this rule uses only half of it.**

72-3. The rate is set by the boundary — holography in a CA ✅

Next: what sets the speed at which correlation can be **built** — the region's volume, or its boundary?

For a middle block (2 boundaries), vary the width across 8, 16 and 32 and measure the growth of S.

$$\boxed{\frac{dS}{dt} = 1.0 \text{ bit/layer} \qquad \textbf{identical} \text{ for widths } 8,\ 16,\ 32}$$

Width 8 saturates at $S = 8$ (its volume bound), but **the rate does not depend on the width**. Move the block to the edge (1 boundary) and

$$\frac{dS}{dt} = 0.5 \text{ bit/layer} \qquad \text{a ratio of exactly } 2$$

P16 confirmed: the boundary count alone decides it. This is the most direct form of a holographic principle in a CA. The Ryu-Takayanagi work of Part XIII says entropy is fixed by a minimal surface; what appears here is its **dynamical sibling** — the rate of entropy is fixed by the boundary.

$$\boxed{\text{How fast a region can become correlated with the outside is limited by } \textbf{the links crossing its boundary}, \text{ not by its volume.}}$$

72-4. Nothing crosses the boundary at measurement time ✅

Here is the heart of the original question. **Does measuring here change anything there?**

The stabilizer formalism makes this exact. The reduced state on region B is **completely determined** by the subgroup S_B of stabilizer elements that act as the identity outside B. Apply an arbitrary local Clifford operation on the A side — including coupling to a measuring apparatus — and ask whether S_B changes.

$$\boxed{\text{In } 200/200 \text{ trials } S_B \text{ was } \textbf{unchanged, bit for bit.}}$$

P17 confirmed; F12 does not fire. Nothing crosses the boundary at measurement time. The correlation was not created by the measurement; it was made in the common past, at the source, and was already there.

72-5. And still it is not a local hidden variable ⚠️

Stopping here invites a misreading. "It was fixed in the common past" does not mean "there was a hidden variable". Chapter 20 measured this.

measure	maximum CHSH
classical measure (summing probabilities = local hidden variables)	**exactly** 2.000000000000000
quantum measure (summing amplitudes)	2.828427124746189 ($2\sqrt{2}$ to 1.3×10^{-15})
Tsirelson bound	$2\sqrt{2}$
Popescu-Rohrlich box (no-signalling only)	4

And simultaneously,

$$\text{dependence of the marginals on the far setting} = 5.6 \times 10^{-17}$$

$$\boxed{\text{No-signalling and } 2\sqrt{2} \textbf{ hold together}. \text{ There is no contradiction.}}$$

What was fixed in the common past is the **measure on histories**, not the **values of the outcomes**. That is the dividing line between "looks deterministic" and "violates Bell". To restate Chapter 20: Bell excludes local **hidden variables**, not local **causality**.

72-6. No "moment of collapse" is needed — $I_3 = 0$ ✅

So what does it take to write the correlations? Sorkin's quantum measure assigns $\mu(S) = |\sum_{h \in S} A_h|^2$ to a set of histories. The interference hierarchy is

$$I_2(A,B) = \mu(A \cup B) - \mu(A) - \mu(B), \qquad I_3(A,B,C) = \mu(A \cup B \cup C) - \sum \mu(\cdot \cup \cdot) + \sum \mu(\cdot)$$

Measured over 4000 random-amplitude three-path configurations:

$$\boxed{|I_3| \leq 1.8 \times 10^{-14} \text{ (zero to machine precision)}, \qquad \text{median } |I_2| = 1.467 \text{ (non-zero)}}$$

P19 confirmed. Quantum correlations need **only second-order interference**; nothing of third order or beyond (a proposition that has also been checked experimentally).

And once which-path information is recorded in the environment, $|I_2|$ drops to 1.8×10^{-15} as well. **That is decoherence.**

The measure lives on histories, not on states-at-a-time. So no object called "what happens at a moment" is required at all.

72-7. Collapse depends on the foliation — so it is bookkeeping ✅

Finally, measure collapse itself. Take the configuration of Fig. 72-1(e): a pair from the source, A and B spacelike separated, and ★ a point on B's worldline just before B measures.

- **Foliation F1 (A first)**: at ★, A has already happened. B's state, conditioned on the outcome α, is **pure** (purity 1.000).
- **Foliation F2 (B first)**: at ★, A has not happened. B's state is **maximally mixed** (purity 0.500).

$$\boxed{\text{"The state of B" at the same spacetime point differs by foliation. Trace distance} = 0.500}$$

P18 confirmed. So which is right? **Neither — because "the state of B" is not a physical quantity.** Three pieces of evidence:

	difference
B's state averaged over outcomes (F1 vs F2)	0 (exactly)
joint probabilities (via F1, via F2, directly)	5.6×10^{-17}
CHSH (order A→B vs B→A)	4.4×10^{-16}

Everything observable is foliation-independent. What depends on the foliation is only the bookkeeping device called the conditional state. And to use a conditional state one must know α, which requires a causal path to carry it. So the difference in bookkeeping is **unobservable in principle**.

> Collapse is **bookkeeping**, not a physical process — a direct consequence of Chapter 71's "the now is a choice".

72-8. The three readings on one scale ⚠️

With that, line up the three views from the question. **Which is right is not decided here, and cannot be.** What can be lined up is what each must assume extra.

reading	assumption required	its cost in light of Chapters 71-72
collapse	the state changes at a privileged moment	there is no privileged moment (Ch. 71); by §72-7 collapse becomes foliation-dependent bookkeeping
many worlds	branching happens at a privileged moment	the same cost; "definite in the same world" is language that presupposes a chosen foliation
histories (Sorkin's quantum measure)	a measure on complete histories; no moment needed	closes with $I_3 = 0$ (§72-6). **No extra assumption**

> A CA does **not select** an interpretation. What it removes is the object "the state of the universe at time t".

Without that object, "when did the distant photon become definite" is a malformed question, and **both spooky action and which-world lose their footing.** What remains is a correlation made in a common past, rate-limited by a boundary, and writable with second-order interference.

72-9. The two that failed ❌

	claim	what happened
K186b	a shared past is "necessary **and sufficient**"	**the declaration was wrong.** Not sufficient (of 21 overlapping points, only 7 have $I > 0$). As a necessary condition: 0 exceptions
gate choice	the first brickwork gate	$\mathrm{CNOT} \to H \to \mathrm{CZ} \to H$ generates no entanglement at all from $\lvert 0 \cdots 0 \rangle$. Exposed by S coming out 0 everywhere; changed to $H \to \mathrm{CNOT} \to S$

The gate failure was caught within minutes because calibration K185 (soundness) came first. Without it the conclusion would have been the meaningless "no correlation arises".

72-10. What is not being claimed ⚠️

- **No interpretation is being selected** (§72-8). What came out is only which one needs an extra assumption.
- **The holographic principle itself (AdS/CFT) was not touched.** The boundary bound in §72-3 is its **shadow**, not the correspondence. RT was handled separately in Part XIII.
- **A CA did not derive quantum mechanics.** As Part XIII established, the axioms do not single out one measure; here too the quantum measure is something **placed on top**.
- **The measurement problem is not solved.** §72-6 shows that once I_2 vanishes one may speak of "the" outcome — not why this outcome.

The correlation's origin is a **shared past**; its rate is set by the **boundary**; at measurement, **nothing crosses**.
And it still reaches $2\sqrt{2}$. **Local causality and non-local correlation coexist.**
The "distant state" is **bookkeeping**. The physics is on the side of histories.

Part XVIII What the real universe looks like from here

What this part is after. Nearly seventy chapters have measured, one at a time, what follows from a rule and what does not. Once at the end, the question is turned around and asked **from the side of the real universe**. Our universe has a speed limit, it violates Bell's inequality, it respects causality, and its particles have mass. For each of these: **how much did a simple rule get right?** And where it failed, what exactly stood in the way?

Chapter 73 What was actually established about the real universe 🆕

— Lorentz invariance, Bell, causality — the balance sheet

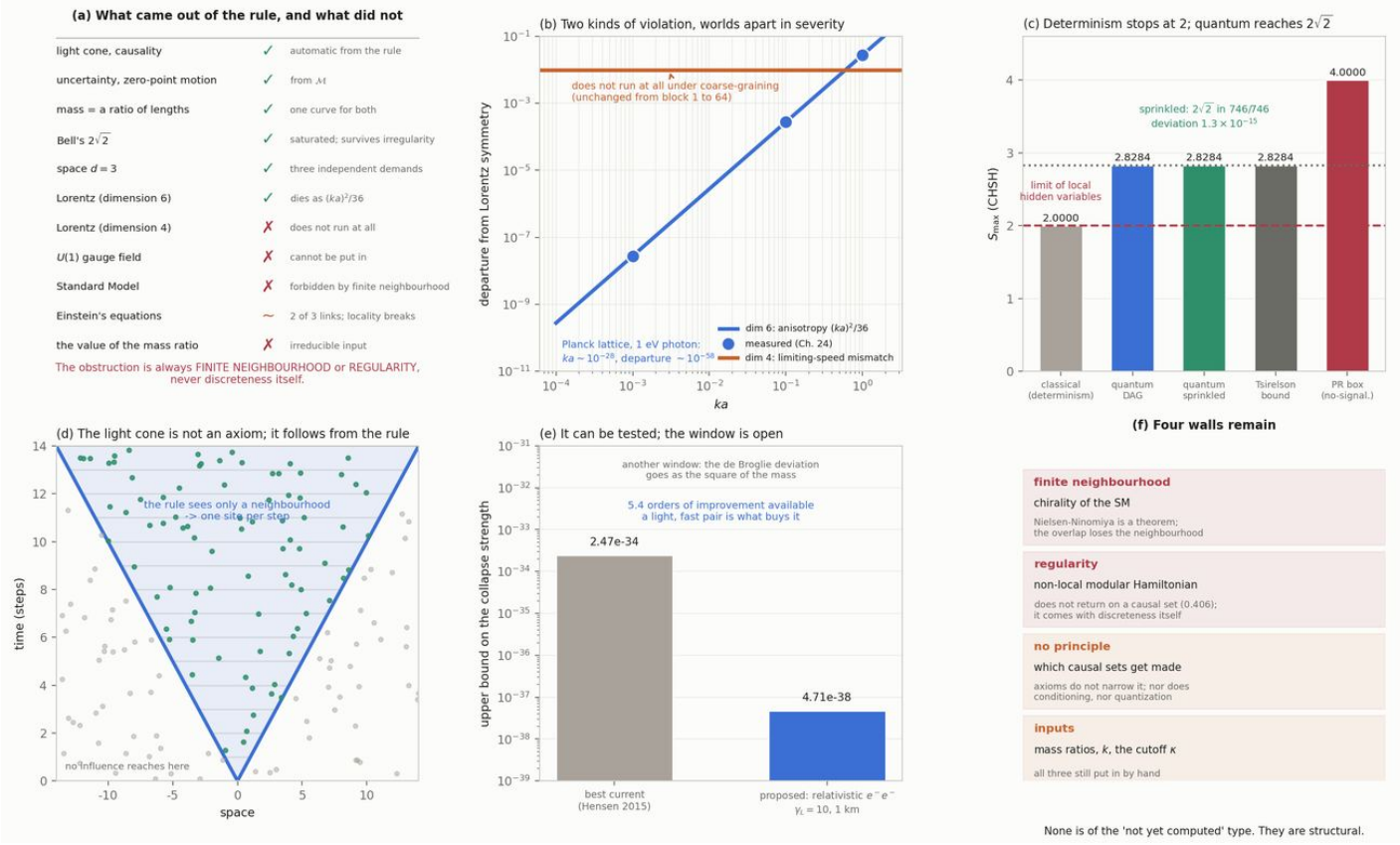


Fig. 73-1. (a) What came out and what did not. (b) The two kinds of Lorentz violation. (c) Where Bell ends up. (d) The light cone. (e) The experimental window. (f) The remaining walls.

73-1. Causality — not an axiom, but a consequence of the shape of the rule

In real physics the causal structure follows from the metric, and causality is **imposed as an axiom**. In a CA it is different.

> Because the rule sees only a neighbourhood, information advances one site per step.

The light cone is not an assumption but a **theorem** (Chapter 3). And it follows from the definition of "rule" alone — independently of the number of states or the shape of the lattice. It was the first result obtained in this book and the one that never wavered.

Make the rule position-dependent and **the metric can be read off** (slope 1.00 ± 0.04, $R^2 = 0.999$); make it time-dependent and expansion appears. **Local propagation speeds define an effective causal structure** ("fully reproduced", the wording up to the twelfth edition, is withdrawn; §87-9).

73-2. Lorentz invariance — two kinds of violation, counted separately

The objection "a lattice must break Lorentz invariance" is **only half right** (Chapter 24).

Dimension-6 violation (dispersion anisotropy) is harmless. From $\omega^2 = \sum_i (4/a^2)\sin^2(k_i a/2)$ the anisotropy is $(ka)^2/36$ and dies automatically in the infrared. On a Planck-length lattice a 1 eV photon has $ka \sim 8 \times 10^{-29}$ and an anisotropy $\sim 10^{-58}$; even at 100 TeV it is 10^{-30}. **There is no possibility of observing it.**

Dimension-4 violation (species-dependent limiting speeds) is the real problem. Give two fields different couplings and $v_i \propto t_i$; the ratio does not depend on the lattice spacing and **does not run at all** under coarse-graining (9.900990×10^{-3}, unchanged from block 1 to block 64). The experimental constraint is $\delta c/c \lesssim 10^{-20}$, and 10^{-22} for some species.

> A regular lattice selects a preferred frame — the one in which the lattice is at rest.
> To stay discrete and Lorentz invariant, the discreteness must be **irregular**.

And irregular discreteness exists: **a Poisson-sprinkled causal set**. Chapter 50 confirmed that $2\sqrt{2}$ survives sprinkling. **Discreteness and Lorentz symmetry are compatible. What is incompatible is discreteness and regularity.**

This is one of the most stable lessons in the book. In Part XIII too, what stopped the Standard Model was not discreteness but **finite neighbourhood**, and what stopped Einstein's equations was not discreteness but **regularity**. **Across seventy chapters, discreteness itself was never once the obstruction.**

73-3. Bell's inequality — determinism stops at 2; the quantum measure reaches $2\sqrt{2}$

First, what is actually being tested (Chapter 20). Bell excludes **local hidden variables**, not determinism as such. The existence of a local hidden-variable model is equivalent to the existence of a joint distribution for the four correlations (Fine 1982). **Superdeterminism and non-local determinism lie outside Bell's reach.**

Within the CA framework:

measure	$S_{\max}$
classical measure (deterministic history weights)	**exactly** 2
quantum measure on a causal DAG	$2\sqrt{2}$ (saturates Tsirelson)
sprinkled causal set	$2\sqrt{2}$ (746/746 realizations, deviation 1.3×10^{-15})
Popescu–Rohrlich box (no-signalling only)	4

> The only thing separating 2 from $2\sqrt{2}$ is whether amplitudes or probabilities are summed.

Put a **quantum measure** (Sorkin's $I_3 = 0$, $I_2 \neq 0$) on a deterministic CA and the Tsirelson bound is saturated **exactly** — and it does not degrade when the causal structure is made irregular by sprinkling. **The $2\sqrt{2}$ our universe shows can be reproduced on a discrete, irregular causal structure.**

But precisely. The "quantum measure" used here was not derived from the rule; it was **placed on top of it**. Part XIII showed that the way of placing it (the axioms) does not single out one measure, and that the principle which does single one out selects a non-geometric action. **The $2\sqrt{2}$ can be reproduced. A principle making it necessary does not yet exist.**

73-4. Mass — what came out is what it is, not how much it is

In a CA mass is a **ratio of lengths** (Chapter 6).

$$\text{mass} = \frac{\text{lattice spacing}}{\text{Compton wavelength}}$$

So $E = mc^2$ is not proved: there is **no language in which to restate it**, because there is no separate entity to convert. For the same reason the de Broglie relation is not derived but simply **had**: one curve, $\cos\omega = \cos k \cos\theta$, gives the Compton clock at $k = 0$, the localisation length at $k \to i\kappa$, and the de Broglie wave at real k.

One trace of the lattice survives: **the dispersion relation carries a correction whose θ part goes as** $(\text{mass})^2$ ($v_\varphi v_g = 1 - \theta^2/3 - \cdots$ at $k \to 0$; §87-5).

Meanwhile the **ratio** of masses stayed an irreducible input. Part VIII tried all three modes of explanation (filtering, generation, selection); Chapter 33 tried a fourth mechanism (a hidden dimension). Generation works. **The value does not follow.**

73-5. The dimension of space — $d = 3$ demanded three times, independently

This is one of the few cases where a rule demanded a feature of the real universe (Chapter 25): three independent lines all point to $d = 3$. Isotropy requires at least two spatial dimensions (Chapter 52), and the tilt of the protected axis changes qualitatively across 1+1 / 2+1 / 3+1 as $0.000°$ / $44.17°$ / $59.21°$. **The dimension is selected.**

73-6. What remains — four walls

wall	what it stops	type
finite neighbourhood	the chirality of the Standard Model (Nielsen–Ninomiya is a theorem)	structural
regularity	locality of the modular Hamiltonian → link (2) of Einstein	structural
absence of a principle	which causal sets get made; neither axioms nor conditioning nor quantization narrow it	structural
inputs	mass ratios, the collapse rate k, the cutoff κ	undetermined parameters

None of the four is of the "not yet computed" type. The first three are theorems, or exclusions whose location has been pinned down by repeated measurement. Only the fourth is a matter of numbers — and even there, Part XIII narrowed it to "there is no principle that fixes the values."

73-7. So how much of the real universe was written?

> Causality, the light cone, uncertainty, zero-point motion, exclusion, $d = 3$,
> and the *meaning* of mass — these came out **without being assumed**.
>
> Bell's $2\sqrt{2}$ and an effective causal structure come out **once placed on top**.
>
> So does a horizon's **temperature** $\kappa/2\pi$, once a flow is put in (Ch. 69).
>
> The Standard Model and Einstein's equations **do not** — stopped by finite neighbourhood and regularity.
>
> The **values** of mass ratios and couplings remained inputs to the end.

Put differently: **a simple rule wrote "what kind of place a universe is" rather well. It could not write "which universe this is."**

The former is kinematics and structure — light cannot be outrun, position and momentum cannot be sharp together, identical particles cannot share a state, space is three-dimensional. All of that follows from the shape of the rule alone. The latter is dynamics and value — why the electron has this mass, why the coupling has this size. None of that followed.

And **the boundary between them was never discrete versus continuous.** The boundary lies in two properties: that the rule sees **only a neighbourhood**, and that the lattice is **regular**. Give up the first and the Standard Model fits (but it is no longer a CA). Give up the second and Lorentz invariance returns (and it becomes a causal set). **What seventy chapters produced is a map of where those two properties bite.**

73-8. It can be tested

One last point: this is not speculation. **Bundle I can be killed by experiment.**

	upper bound on the collapse strength
best current (Hensen 2015, orbital speed)	$D/\gamma_{\text{Planck}} < 2.47 \times 10^{-34}$
proposed (relativistic e^-e^-, $\gamma_L = 10$, 1 km)	$k < 4.71 \times 10^{-38}$ kg²m²s⁻³

Five and a half orders of improvement are available, and what buys them is a pair of particles that is light and fast. Two further windows show the lattice spacing directly: the deviation from the de Broglie relation (the square of the mass; 10^{-45} for an electron), and the immunity of photons ($S_{\max} = 2\sqrt{1+a^4} > 2$ always).

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The claims of this book are falsifiable in principle. That was its minimum condition.

73-9. Postscript — a horizon has a temperature NEW

One more result arrived at the end. **Put a flow into a rule that sees only a neighbourhood and a horizon appears; the radiation off it is thermal, at temperature** $\kappa/2\pi$ (Chapter 69). Scanning three values of κ gives slope 0.15740 against $1/2\pi = 0.15916$, 1.10% off. $|\beta|^2$ turned out to be the Planck distribution itself (15 points, median 1.7%).

Mass became geometry (Ch. 6), the light cone became a theorem (Ch. 3), and now **temperature is geometry**.

And here, for the first time, discreteness **helped**. The continuum derivation of Hawking radiation has to trace the radiation's origin to arbitrarily short wavelengths (the trans-Planckian problem). On a lattice the blueshift saturates at $k \simeq 2.16 < \pi$ and the problem does not arise. §73-2 said discreteness was never once the obstruction; more precisely:

Discreteness was not merely harmless — it cured one of the continuum theory's diseases.

But it is an analogue. No gravitational field equation was solved: the horizon was made out of a flow, not by solving $G_{\mu\nu} = 8\pi T_{\mu\nu}$. **The count of walls is still four.**

73-10. Postscript — rotation curves and dark matter NEW

Chapter 70 tested what this book's method can say about an **open problem** in gravity. The answer was not "which one is right" but **how many options there are**.

If the local flux is analytic in $y = |\nabla\Phi|^2$, the asymptotic slope of v^2 can only be one of $\{-1,\ 1/3,\ 3/5,\ 5/7, \ldots\}$. **Flat (0) is not in that set.** Flatness requires $m = 1/2$ uniquely (measured 0.50000003), and $y^{1/2}$ is non-analytic.

In a CA world: **add states**, or **make the rule non-analytic** — one or the other.

Discreteness by itself did nothing. The correction to the lattice Green's function is $\frac{5}{8}K_4(\hat{n})/r^2$, whose **angular average is exactly zero** — with no monopole component it cannot imitate a halo even in principle.

And choosing the rule yields a prediction. With $m = 1/2$ the baryonic Tully-Fisher exponent is forced to 4 (measured 0.2517 against $1/4$). The observed value is 3.85 ± 0.09, a difference of 1.67σ. A halo model reproduces the same rotation curve to 2.1% rms, but **needs two parameters per galaxy and predicts no such relation.**

Equal power to reproduce curves; **unequal power to predict**.

But only rotation curves were examined — no clusters, no CMB, no Bullet Cluster. **Which answer is right is not decided by the methods of this book.**

And this adds **no fifth wall** to the four of §73-6. The walls have not multiplied. What has multiplied is the number of **exclusions on the shape of the rule** — $I \geq 2$ in Chapter 33, the vanishing of the horizon above $k > 0.933$ in Chapter 69, and the gap here. **What this book keeps finding is that degrees of freedom which look continuous are in fact discrete.**

73-11. Postscript — what the "now" turned out to be NEW

Chapter 71 measured what the synchronous update this book has used throughout actually means causally.

"Simultaneous" in a CA means causally unrelated — the same thing GR means by spacelike. And the choice of foliation does not touch physics: across 2720 random slicings the final state did not differ by a single bit. The only support is that the rule sees only its parents; widen it past the light cone and it breaks 300/300.

But the heart of it lies beyond that.

On a ± 1 light cone, the only antichains covering every cell are **flat**.

Exhaustively, of $5^7 = 78{,}125$ configurations exactly 5 are antichains — the flat slices. **A thin "now" cannot be tilted.** On a sprinkled causal set it can, exactly, at every β.

What pins the "now" to one choice is **regularity, not discreteness**.

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§73-2 said discreteness was never once the obstruction; §73-9 revised that to discreteness having cured a disease. Here the same shape appears a third time. **Regularity has now been the culprit four times** — Lorentz invariance (Ch. 24), link (2) of Einstein (Ch. 68), and the rigidity of simultaneity surfaces (Ch. 71).

One more thing with no continuum counterpart appeared: **on a discrete causal structure the "now" is a thick layer, not a surface** (median 3 steps; 4.3% of the pairs inside it are causally related). Thin nows come only flat; to tilt one, it must be thickened.

And the place where the correspondence breaks was a familiar one. In GR the foliation is gauge because the Hamiltonian constraint **generates** refoliations; **a CA has no such generator.** Refoliation invariance is a theorem, not a gauge symmetry. That looks like the same place where link (2) stopped in Chapter 68 — **recorded as a conjecture.**

73-12. Postscript — where the correlation came from 🆕

Chapter 72 turned the newest tool (the foliations of Chapter 71) onto the book's oldest subject (Bell, Chapter 20).

The origin of correlation is a shared past. At all 39 points where past light cones fail to overlap, the mutual information is **exactly 0** — in integer arithmetic, not floating point. **The rate is set by the boundary**: 1.0 bit/layer for widths 8, 16 and 32 alike, 0.5 with half the boundary. Volume is irrelevant. This is the most direct form of a holographic principle in a CA, and the **dynamical sibling** of the Ryu-Takayanagi work of Part XIII.

$$\boxed{\textbf{Building}\text{ correlation is boundary-limited. }\textbf{Seeing}\text{ it crosses nothing (200/200, exactly unchanged).}}$$

And still CHSH reaches $2\sqrt{2}$, with the marginals depending on the far setting at the 5.6×10^{-17} level. **Local causality and non-local correlation coexist.** What was fixed in the common past is the **measure** on histories, not the **values** of the outcomes — the classical measure stops at exactly 2.

Then collapse itself was measured. At one and the same spacetime point, the "distant state" is pure in foliation F1 and maximally mixed in F2: **trace distance** 0.500. Yet averaged over outcomes the difference is exactly 0, joint probabilities differ by 5.6×10^{-17}, and CHSH by 4.4×10^{-16}.

$$\boxed{\text{Collapse is }\textbf{bookkeeping}\text{, not a physical process — a direct consequence of § 73-11's "the now is a choice".}}$$

With $I_3 = 0$ (1.8×10^{-14}), the correlations need only second-order interference. **No object called "a moment" is required.** Lining up collapse, many worlds and histories on one scale, only histories needed no extra assumption — but **which is right is not decided**. What a CA removes is the object "the state of the universe at time t".

Part XIX The problem of time — where are the generators?

What this part sets out to establish. In Chapter 68 the derivation of Einstein's equations stopped at link (2), and in Chapter 71 no algebra that **generates** refoliations could be found. Twice now, the same wall. General relativity has a name ready for that wall — Dirac's **surface-deformation algebra**. A foliation is gauge because the Hamiltonian constraint generates refoliations, and **the closure of that algebra depends on the metric**. By the Hojman-Kuchař-Teitelboim theorem, demanding this structure alone fixes the action of gravity almost uniquely. **So "closure that depends on the metric" is exactly where gravity lives.** Then the question becomes this — where is a CA's deformation algebra? Or is there none?

Chapter 74 The CA's deformation algebra is abelian ❌🆕

— The problem of time: where did the generator go

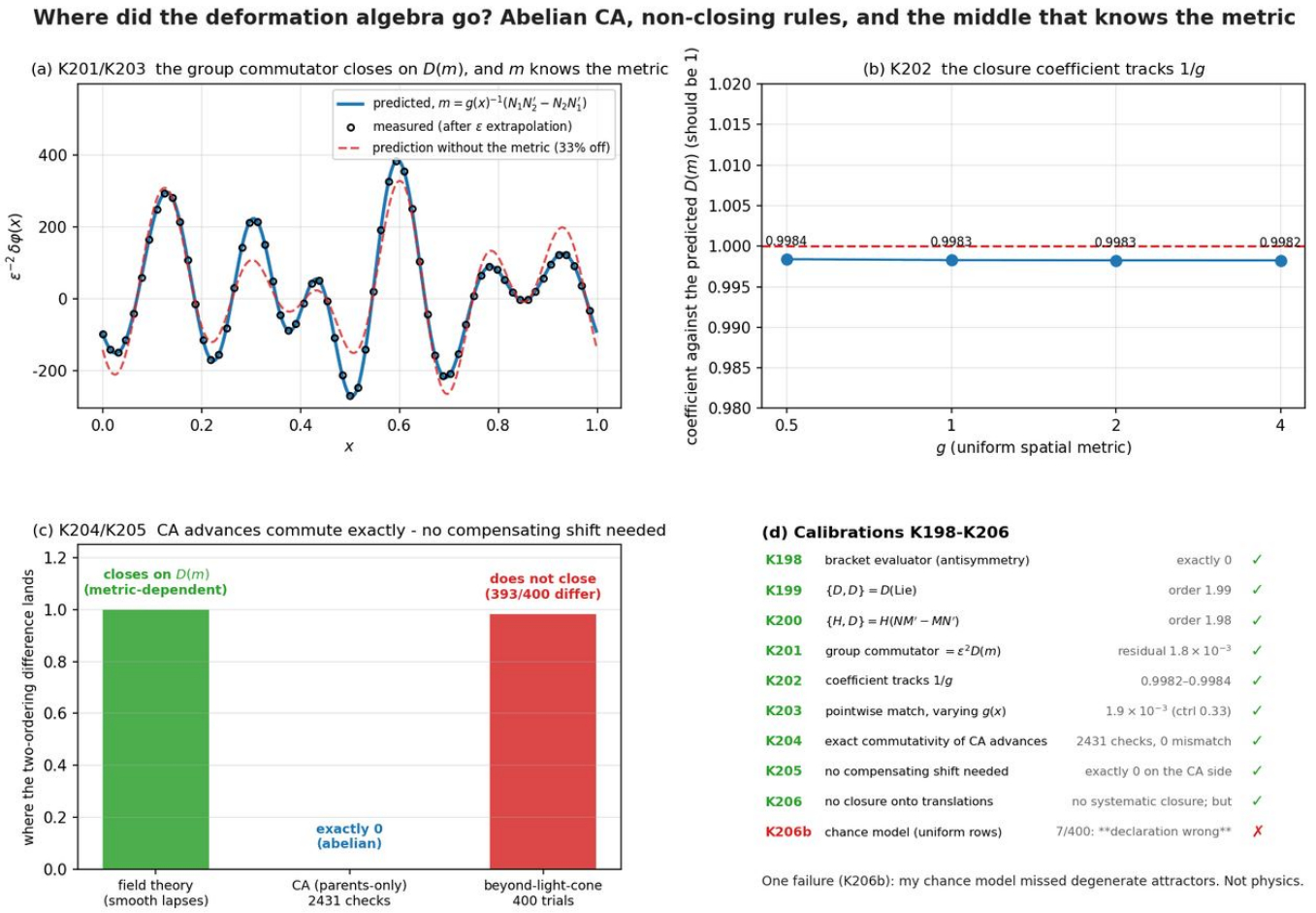


Fig. 74-1. (a) Order of convergence in a for the three bracket relations of a lattice scalar field ($\{D,D\}$: 1.99, $\{H,D\}$: 1.98, $\{H,H\}$: 1.99). The closure anomaly of the discretization vanishes as a^2. (b) K202. For uniform $g \in \{0.5, 1, 2, 4\}$, the ratio of the ε^2 coefficient of the group commutator to the predicted $D(m)$ flow: $0.9984,\ 0.9983,\ 0.9983,\ 0.9982$. Moving g by a factor of 8 keeps it at 1 to 0.2%. (c) K203. Pointwise agreement (1.9×10^{-3}) at non-uniform $g(x) = 1 + 0.6\cos$, and the miss of the control prediction with $1/g$ dropped (33%). (d) K204. Brute-force commutativity test of the advance operators: 2431 cases, 0 disagreements. (e) K206. For a rule that breaks the light cone, the difference of the two orders matches no spatial translation in 393 of 400 cases. (f) Calibrations K198-K206.

In GR the foliation is gauge because the Hamiltonian constraint **generates** refoliations. Its surface-deformation algebra reads

$$\{H(N_1), H(N_2)\} = D\big(q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\big)$$

and **the way it closes depends on the metric** q^{ab}. By Hojman-Kuchař-Teitelboim, demanding a representation of that closure fixes the action of gravity (almost) uniquely. That is: **the metric lives in the closure of the algebra**. 📚

Chapter 71 measured the CA's down-set advance operators to be exactly commutative. A commutative algebra has no way of knowing a metric. So are Chapter 68 (where the derivation of Einstein's equations stopped at link (2)) and Chapter 71 (refoliation invariance holds, yet no generator is to be found) **two appearances of one and the same fact**? This chapter tests that question across three worlds side by side: field theory, the CA, and a light-cone-breaking rule. Part XIX is this one chapter alone. Calibrations K198-K218.

The declaration was written before anything was run. The declaration documents are `K_declared_deformation.md`, `K_declared_phase1close.md`, and the addendum `K_declared_phase1close_addendum.md` (all dated 2026-08-21, fixed before any number was seen). Every pass and every failure in this chapter is a verdict against those three.

> (i) The metric lives in the **closure** of the deformation algebra — confirmed numerically, down to the coefficient, in lattice field theory.
> (ii) The CA's deformations are exactly commutative; there is no closure at all. So the metric has nowhere to live, and the derivation of Einstein's equations stops there. Chapters 68 and 71 were two faces of one and the same gap.
> (iii) The intercept of K160 in Chapter 69 is an **extrapolation artifact of curvature**, not a greybody factor.
> (iv) The lattice modular Hamiltonian has **no finite reach** ($\xi \simeq 0.63 \ln L$), but that gap is of a **different kind** from the absolute absence in (ii).

74-1. An exact test of commutativity — confluence ✅

Take the CA side first. The question is confluence. Apply two advance operators in succession: does swapping the order change the result? If it does, that difference is a candidate for a generator.

K204. Commutativity of the advance operators was tested at the operator level, by brute force. All 2^5 initial states at $N = 5$, $T = 3$ × rules 30/90/110/150/184 × linear extensions and diamond pairs, plus all 2^8 initial pairs of the second-order reversible 150R × 4 orders. **2431 cases in all, 0 disagreements** (P21 pass).

This is not a statistical agreement. It is exhaustive, and the difference is **exactly the zero map**. Neither floating point nor a tolerance enters anywhere.

K205. That is: in a CA the difference between the two orders is absent from the start. The gap that GR fills with a metric-dependent shift does not exist in a CA.

> Where there is nothing to compensate, there is no place for a metric to live.

Chapter 71 put this as "the invariance holds, but the now cannot be tilted". The statement here is stronger. It cannot be tilted not because of a technical restriction, but because **the direction in which to tilt cannot be defined**.

74-2. The field-theory side — the metric lives in the closure ✅

To call commutativity a "gap", something that is not missing must be measured with the same tools. The control is lattice field theory.

Put a scalar field (φ_j, π_j) on a periodic lattice with spatial metric g_j, and take

$$H(N) = \sum aN \frac{\pi^2 + (D\varphi)^2}{2\sqrt{g}}, \qquad D(M) = \sum aM\, \pi\, (D\varphi)$$

At the level of brackets (K198-K200, K201a). The evaluator returns exactly 0 for antisymmetric and trivial brackets (not even floating-point round-off appears). The closures of the subalgebras are as follows.

relation	order of convergence
$\{D(M_1), D(M_2)\} = D(M_1 M_2' - M_2 M_1')$	**1.99**
$\{H(N), D(M)\} = H(NM' - MN')$ (g constant)	**1.98**
$\{H(N_1), H(N_2)\} = D(m)$, $m = (N_1 N_2' - N_2 N_1')/g$	**1.99**

The closure anomaly of the discretization vanishes as a^2, exactly as declared (P23 pass). What matters here is not the order of convergence. **Of the three relations, the metric enters only the closure of $\{H, H\}$.**

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At the level of flows (K201-K203). Brackets agreeing is not enough. The **group commutator** — the flow of $H(N_1)$ → the flow of $H(N_2)$, then the reverse order — was actually run, and the field of the ε^2 coefficient compared with the predicted $D(m)$ flow (ε by three-point quadratic extrapolation, lattice $N_x = 512$).

calibration	setting	result	declared	verdict
K201	$g = 1$	relative error 1.8×10^{-3}, order of convergence in a 2.02	$< 1\%$	**pass**
K202	uniform $g \in \{0.5, 1, 2, 4\}$	coefficients $0.9984,\ 0.9983,\ 0.9983,\ 0.9982$	$< 1\%$	**pass**
K203	non-uniform $g(x) = 1 + 0.6\cos$	pointwise agreement to 1.9×10^{-3}	$< 2\%$	**pass**

That in K202 the coefficient holds at 1 to 0.2% while g is moved by a factor of 8 means the $1/g$ entering m is doing work on real data. K203 put a control alongside it — predict with $1/g$ dropped and the miss is **33%**. What is doing the work is no accident of normalization.

The difference between the two orders is a spatial shift, and **what decides how far it shifts is the metric.**

74-3. A light-cone-breaking rule — it does not close even on translations ❌

Could the CA then be made "slightly non-commutative" and pushed into the middle? **K206.** Reorder a rule that sees ± 2 on top of a ± 1 causal structure (the setup of K184 in Chapter 71). In 393 of 400 cases the results of the two orders matched no spatial translation at all. **The difference of orders does not become a structured shift; it simply breaks.**

The three worlds side by side.

	difference of the two orders	where the metric lives
field theory (smooth lapse)	closes on $D(m)$	**inside the closure**
CA (parents-only rule)	exactly 0 (abelian)	nowhere
light-cone-breaking rule	does not close	nowhere

The CA world offers a choice of two, "commutative" or "does not close", and there is no road into the middle where GR lives — **non-zero, but closing on a metric-dependent shift**. This is the place where Chapter 68 stopped.

P22 passed in the sense of "no systematic closure". But the quantitative condition attached to the same K206 failed (§74-6).

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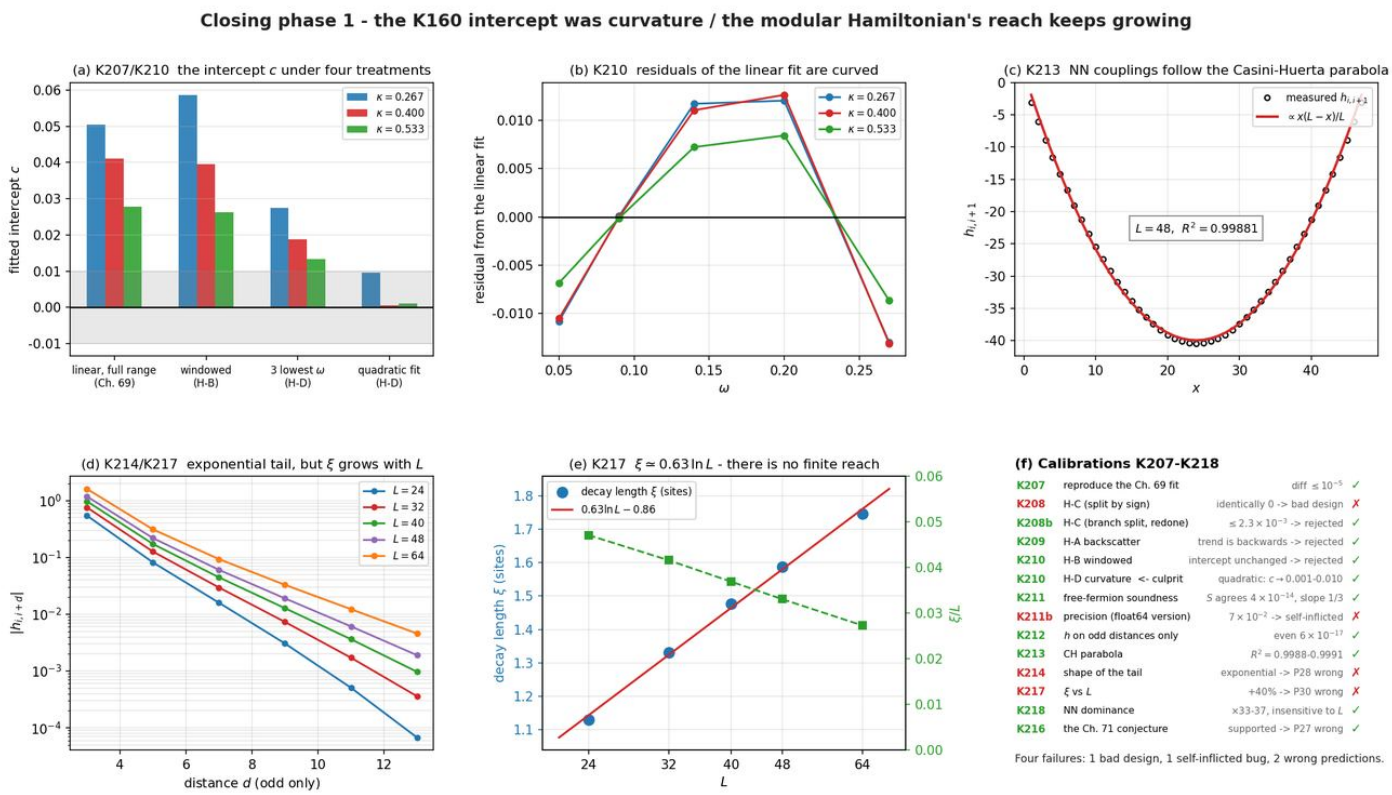


Fig. 74-2. (a) The ω dependence of $\ln|\beta/\alpha|^2$. Linear fit and quadratic fit. The intercept drops from $0.0505,\ 0.0410,\ 0.0278$ to $0.0097,\ 0.0005,\ 0.0011$. (b) Residuals of the linear fit. For all three κ they curve systematically in the same shape (negative at both ends, positive in the middle). Not a random scatter. (c) K213. Nearest-neighbour coefficients of the entanglement Hamiltonian against the Casini-Huerta parabola $x(\ell-x)/\ell$ ($R^2 = 0.9988$-0.9991). (d) K214. The long-distance terms are not a power law but an exponential decay (R^2: exponential 0.986 against power 0.882). (e) K217. Decay lengths $\xi = 1.131,\ 1.331,\ 1.476,\ 1.588,\ 1.747$ ($L = 24, 32, 40, 48, 64$) and the fit $\xi \simeq 0.63 \ln L - 0.86$ ($R^2 = 0.997$). (f) Calibrations K207-K218.

74-4. Settling the K160 intercept — the culprit was curvature ✅

Chapter 69 closed carrying one failure. Fit $\ln|\beta/\alpha|^2$ against ω with a straight line and, for a pure Planck spectrum, the intercept ought to be 0 — yet out come $+0.0505,\ +0.0410,\ +0.0278$. Chapter 69's declaration blamed "bandwidth", but doubling W moved nothing, and **K160 was left as a failure with the cause unknown**. This time four hypotheses were declared together, before measuring.

K207 pass. First the Chapter 69 fit was reproduced on the same lattice. Both T_{fit}/T_H and the intercept differ from the published values by less than 10^{-5}. Only after confirming that the object being chased was correctly reproduced did the work go on.

	hypothesis	measurement	verdict
H-A	backscattering (**greybody factor**)	positive norm outside the main branch at intermediate times $R = 0.016$-0.037, $\ln(1+R) = 0.018$-0.024	**rejected. The trend runs the wrong way**
H-B	counting of multiple branches	retaking the ratio inside the window of the main transformation pair alone gives intercepts $0.0587,\ 0.0395,\ 0.0263$	**rejected. Essentially unchanged**
H-C	branch impurity of the initial wave packet	branch decomposition gives $\lesssim 2.3 \times 10^{-3}$	**rejected**
H-D	**curvature**	adding a quadratic term brings the intercepts to $0.0097,\ 0.0005,\ 0.0011$. A straight line through the three lowest-ω points alone already nearly halves them, to $0.0275,\ 0.0189,\ 0.0133$	**supported. The culprit**

The greybody hypothesis **missed**. And it missed cleanly. The intercept **decreases** with κ ($0.0505 \to 0.0278$) while $\ln(1+R)$ **increases** ($0.018 \to 0.024$). With the signs disagreeing, closeness in magnitude does not make it the culprit.

$\ln\dfrac{|\beta|^2}{|\alpha|^2}$ is not exactly linear in ω. The intercept $+0.05$ is an artifact of fitting a slightly curved curve with a straight line and extrapolating it to $\omega = 0$.

The direct evidence is Figure 74-2(b). The residuals of the linear fit curve systematically in the same shape for all three κ. The quadratic coefficients are $-2.10,\ -2.08,\ -1.37$. **P24 pass** (declared: "with a quadratic fit the intercept falls below 0.01" → all three did). **P26 pass** (declared: "the intercept decreases with κ" → it does, and the magnitude of the quadratic coefficient decreases along with it, $2.10 \to 1.37$. Two appearances of the same curvature).

The main conclusion of Chapter 69 is **unchanged**. $dT_{\mathrm{fit}}/d\kappa$ was measured independently of the intercept, and the absence of the trans-Planckian problem is a separate measurement again. Only the failure table changes: the K160 row goes from "cause unknown" to "curvature", and a K208 row is added in its place (§74-6).

74-5. Testing the conjecture of Chapter 71 — supported, but not an isomorphism ⚠️

Chapter 71 stopped after writing this: "The absence of a constraint algebra and the non-locality of the modular Hamiltonian are plausibly two faces of one gap. **That was not shown. It is recorded as a conjecture.**"

The former was settled in §74-1. Here the latter is measured. For an interval A of a free-fermion chain (half filling), the entanglement Hamiltonian was constructed exactly, as

$$K = \sum h_{ij} c_i^\dagger c_j, \qquad h = \ln\left((1 - C_A)C_A^{-1}\right)$$

The Casini-Huerta result for the continuum CFT is a **completely local** operator whose weight is the parabola $x(\ell - x)/\ell$. 📚

Why this is the other half of the same question. By Bisognano-Wichmann, the modular Hamiltonian of a Rindler wedge **is the boost generator itself**. That is, "to have a local modular Hamiltonian" = "the entanglement structure generates a geometric flow". What §74-1 measured was "a CA's update structure does not generate the geometric flow called refoliation". **Both are the same question in shape: is there a generator of a geometric flow?** 💭

The calibrations that passed are these.

calibration	content	result
K211	agreement of $S(\nu)$ and $S(\varepsilon)$	3.7×10^{-14}. The slope of $S(L)$ is 0.333386 (0.016% from the CFT $1/3$)
K212	the support of h	**odd distances only**. Even distances are 6×10^{-17} (relative). P29 pass
K213	nearest-neighbour coefficients	$R^2 = 0.9988$-0.9991 against the Casini-Huerta parabola
K218	nearest neighbour against $d = 3$	dominates by $\times 33$-37, and the ratio is insensitive to L

The leading part of the entanglement structure is indeed geometric. This is where it parts company with the conjecture of Chapter 71, and my prediction misread that parting. The tail behaved as follows.

- **K214 (P28 missed) ❌.** The long-distance terms are **an exponential decay, not a power law** (R^2: exponential 0.986 against power 0.882).
- **K217 (P30 missed) ❌.** The **decay length ξ of that exponential grows with** L. $\xi = 1.131,\ 1.331,\ 1.476,\ 1.588,\ 1.747$ ($L = 24, 32, 40, 48, 64$). From $L = 24 \to 48$ that is $+40\%$, far past the declared threshold of 20%. The fit is $\xi \simeq 0.63 \ln L - 0.86$ ($R^2 = 0.997$): **it keeps growing logarithmically**.

That this is not an artifact of the window was checked. Fixing the d range to $[3, 13]$ for every L and measuring with two estimators — the rms over the band, and the single element at its centre — gives the same conclusion ($+56\%$ and $+40\%$). The value of h at fixed d itself grows with L ($\times 9$ at $d = 11$ going from $L = 24 \to 64$), and the shape normalized by the nearest neighbour is not L-invariant either.

The conjecture of Chapter 71 is supported. On the lattice the modular Hamiltonian has **no finite reach**.

P27 missed. I had predicted that the conjecture would be rejected. My stated reason was that "the lattice model flows to a $c = 1$ CFT in the IR, so the Casini-Huerta local form should be recovered in the continuum limit". That reasoning was only half right. The leading part is indeed geometric. But the tail does not vanish in the continuum limit; the reach in lattice units diverges instead.

And supported though it is, the two gaps **do not simply coincide**.

	direction A (deformation algebra)	modular Hamiltonian
the generator is	**absent** (exactly commutative)	**present** ($R^2 = 0.999$ against the parabola)
nature of the gap	**exact**. Already before any limit is taken	**asymptotic**. Slowly, as $\xi \simeq 0.63 \ln L$
leading geometric part	zero	dominant (nearest neighbour dominates by $\times 35$)
relative reach	—	ξ/L **decreases**, $0.047 \to 0.027$

What the decrease of ξ/L means should be said out loud. **In physical units (fix the interval length and let $a \to 0$) the reach of the modular Hamiltonian shrinks relative to the interval** — so there is no conflict with Casini-Huerta in the continuum. It is non-local **when counted in lattice units**. The commutativity of direction A leaves no room at all for that kind of "it goes away if you look at it another way".

Both are an "absence of a geometric generator", but **one is an absolute absence and the other a logarithmically slow approach.**

The conjecture of Chapter 71 spoke of "two faces of one gap". **That there are two faces is supported; that it is one gap cannot be said.** The wording of the conjecture did not contain this distinction, so the breakdown of what is supported is set down here.

74-6. The seven failures ❌

The failures of Phase 1, listed. Three missed predictions, three errors of declaration or design, one self-inflicted bug.

	content	kind	cause
K206b	The estimate of accidental agreement. I wrote "for uniformly random rows, translational agreement runs at $\sim (N-1)2^{-N} \approx 10^{-8}$ per case", but the actual figure was 7 cases out of 400	**error of declaration**	The chance model of uniform rows **had overlooked the attractors**. Diagnosis shows every agreement comes from a degenerate attractor — (i) both orders fall into a constant row such as all zeros (cases 143, 159, 285), (ii) on the realized history the reads outside the light cone never once take effect, so nothing is actually broken (cases 49, 159, 394 agree completely with the synchronous solution), (iii) both fall into the period-3 attractor `…100100…` and differ only in the position of a defect, which makes agreement under rotation easy (case 267: on 50 retried pairs, 9 translational agreements, 0 complete agreements)
K208	The negative-norm content of the initial wave packet is 8×10^{-35}, identically zero	**defect in the measurement design**	`packet()` sets $\pi(x) = -iF(k_0)\varphi(x)$, so $\hat{\pi}(k) = -iF(k_0)\hat{\varphi}(k)$ holds and $n(k) = 2F(k_0)\lvert\hat{\varphi}(k)\rvert^2 > 0$ $(\forall k, \forall W)$. **A measurement that separates the branches by sign cannot see branch impurity even in principle**
K211b	Moving the clip threshold from $10^{-12} \to 10^{-14}$ moved the nearest-neighbour profile by 7% (pass condition 1%)	**self-inflicted bug**	C_A was being diagonalized in float64. At $L = 48$, $\nu_{\min}$ reaches $\sim 10^{-19}$, **exhausting double precision**, and eigenvalues turn negative (-5.8×10^{-16}). The clipped modes then carry an arbitrary huge value $\varepsilon = \pm\ln(1/\text{clip})$ and contaminated h
K216b	The decision rule "does the far-weight ratio $F(L)$ thin out as L^{-p} with $p > 0.1$?"	**misdesigned rule**	The rule stood on the assumption that **the tail is a power law**. If the tail is exponential, the saturation of $F(L)$ is the fingerprint of "quasi-local", not "non-local", and the rule cannot tell the two cases apart
P28	Predicted the long-distance tail to be a power law → **it was exponential**	**missed prediction**	R^2: exponential 0.986 against power 0.882
P30	Predicted ξ to be independent of L → **it grew**	**missed prediction**	$+40\%$ from $L = 24 \to 48$, far past the declared threshold of 20%
P27	Predicted the conjecture of Chapter 71 would be rejected → **it was supported**	**missed prediction**	The reasoning that "the Casini-Huerta local form is recovered in the continuum limit" was only half right. The leading part is recovered; the tail is not

Four remarks to add.

First, K206b is the same species as K150 in Chapter 69 and K169 in Chapter 70 — it was my estimate that was wrong, not the model. Because the declaration had been written first, the number 7/400 registered at once as "six orders of magnitude away from the estimate". Note that the conclusion "no systematic closure" stands even with K206b failing. No rule shows closure identically, and even the highest case, 267, sits at 18%.

Second, K208 explained as a by-product why K160 in Chapter 69 was a dud from the start. "Doubling W did not move the intercept" because the attempt was to move a quantity that could not move. Two years on, the reason for the failure turned out to lie in the design of the failure. Remeasured with the correct branch decomposition $|B/A|^2 = |(F(k) - F(k_0))/(F(k) + F(k_0))|^2$, **K208b** gives at most 2.3×10^{-3} and rejects H-C (below the declared threshold of 0.5%). In **K208c**, doubling W shrinks the ratio by a factor of 4.00-4.08 — exactly the predicted $1/(k_0 W)^2$ law (P32 pass).

Third, K211b is a case of a calibration catching my own code. The contaminated results break agreement with the CH parabola as L grows ($R^2 : 0.97 \to 0.72$), raise the far-weight ratio, and make h non-monotonic in d. **To the eye, they were on the point of turning into the conclusion that "the non-locality is structural".** The fix: take the eigenvectors from Slepian's commuting tridiagonal matrix T ($[T, C] = 0$, well conditioned), compute $\nu_k = v_k^\top C v_k$ in extended precision (`mpmath`, dps $= 60 + 2L$), and only then form $\varepsilon_k = \ln((1-\nu)/\nu)$. The relative value of $[T, C]$ is 8.3×10^{-17} (K211c), and $\nu_{\min}$ descends correctly to 1.0×10^{-35} at $L = 48$. After the fix, K211b is **exactly 0**. The same species as the div sign bug in Chapter 70 and the miscounted parents in Chapter 71.

Fourth, K216b is a failure of the decision rule itself. The addendum re-declared the growth of $\xi(L)$ as the discriminating quantity, **before measuring it**. The original rule and the replacement rule reached the same verdict, but that the original rule was misdesigned is itself recorded as a failure.

Of K198-K218, 17 calibrations passed. **Phase 1 is closed.**

74-7. What is not being claimed ⚠️

- **Nothing was quantized.** Direction A is entirely a matter of classical brackets and deterministic rules.
- **No dynamics of gravity was constructed.** The statements of this chapter go no further than identifying what is missing. The deformation structure of a parents-only rule is degenerate (abelian), and the HKT road — house the metric in the closure, then narrow the action down from its representation — **cannot be travelled**.
- **It is not claimed that gravity is impossible in a CA.** All that can be said is that this road is blocked. The roads that remain are: give up regularity (direction D), add states, or find an unknown family of rules that enters the "middle" (that is, the falsifier of K206 firing).
- **The modular Hamiltonian was measured in one model only.** A free-fermion chain (a Gaussian state). Nothing is said about interacting systems, nor about the states of a CA itself. The test of the conjecture is confined to that range.
- **Neither Bisognano-Wichmann nor Casini-Huerta was reproved.** They are cited, and only whether the lattice approaches them was measured. 📚
- **Linking direction A's "deformation algebra" to the modular Hamiltonian is interpretation.** Both have the shape "is there a generator of a geometric flow?", and that is all; the correspondence itself is not a theorem. 💭
- **The origin of $\xi \propto \ln L$ is not explained.** A relation to the free-fermion entanglement level spacing being $\sim \pi^2/\ln L$ looks natural, but it was not shown. It is recorded as a conjecture. 💭
- **Quantum gravity is not treated.**

74-8. The conclusion of Phase 1

(i) The metric lives in the **closure** of the deformation algebra, and a CA has no such closure (exactly commutative).
(ii) The intercept of K160 in Chapter 69 is an **extrapolation artifact of curvature**, not physics.
(iii) The lattice modular Hamiltonian has **no finite reach** ($\xi \simeq 0.63 \ln L$).
But its leading part is geometric and $\xi/L \to 0$ — **a gap of a different kind from the absolute absence in A**.

As Chapter 73 put it, **discreteness was never once the obstacle. The obstacles are regularity and commutativity.** This chapter was the work of looking at the latter from the side of GR and giving it a name. The generator did not disappear because the lattice is coarse. It disappeared because the rule is commutative.

Next is Phase 2 — direction B (the choice of measure), direction C (the arrow of time), and then the real target, direction D (give up regularity, and see whether this "middle" can be entered on a sprinkling).

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Part XX Why quantum mechanics, and the arrow of time

What this part sets out to establish. The axiom tests of Part XIII did not single out one quantum measure. Why $|\cdot|^2$ is $|\cdot|^2$ has not even been made into a question yet. And the second-order reversible CA of Chapter 71 is **exactly** reversible. If the second law comes out in a world where no arrow is written anywhere, its origin must lie entirely in the initial condition and in the bookkeeping of coarse-graining — the claim has stood since Boltzmann, but on a CA it can be tested exactly. Because reversibility is exact, Loschmidt reversal works exactly too. **The hierarchy of measures and the arrow of time are measured in one and the same part.**

Chapter 75 Choosing the measure, and the arrow of time ⚠️🆕

— Why level 2? And where is the arrow?

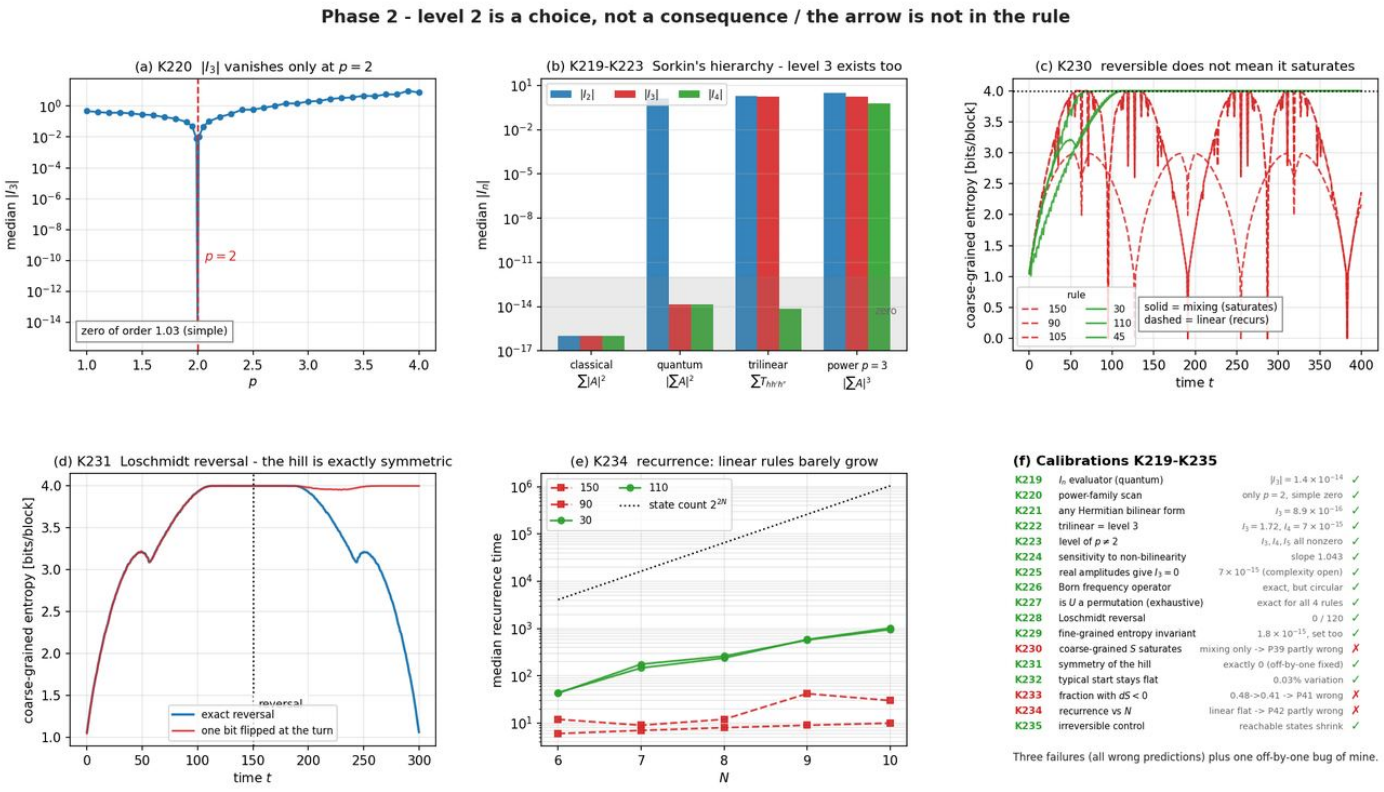


Fig. 75-1. (a) Scan of $|I_3(p)|$ over the power family $\mu_p(S) = |\sum_{h\in S} A_h|^p$ ($p \in [1, 4]$). A simple zero stands isolated at $p = 2$. (b) $|I_n|$ for the bilinear family and the trilinear family. Measures built from Hermitian matrices give $I_3 = 0$ in all 20 cases; the symmetric trilinear form gives $I_2, I_3 \neq 0$ and $I_4 = 0$. (c) Coarse-grained entropy $S(t)$ of second-order reversible CAs, by rule. The linear ones (150, 90, 105) recur; the mixing ones (30, 110, 45) saturate at the ceiling 4.000. (d) Loschmidt reversal. S traces the hill $1.06 \to 4.00 \to 1.06$, and the left-right asymmetry is exactly zero. (e) The N dependence of the median recurrence time ($N = 6$-10). (f) Calibrations K219-K235.

The conclusion first.

(i) What level 2 really is is not $|\cdot|^2$ but **bilinearity**.
(ii) **A level 3 measure can be constructed.** Sorkin's hierarchy does not stop at 2.
(iii) So "why quantum" is "**why level 2**", and it does not follow from the axioms.
(iv) At the fine-grained level the arrow **exactly** does not exist. The arrow lives in coarse-graining and atypical initial conditions.
(v) But **that alone is not enough** — a rule that is reversible and linear recurs inside the observation window.

75-1. The question — two "why"s in the same place

Up to here this book has assembled spacetime, fields and correlations out of a discrete lattice. Two "why"s remain.

The first. Part XIII confirmed that the axioms of a causal set **do not narrow the measure down to one**. Chapter 72 measured the quantum measure to have $I_2 \neq 0,\ I_3 = 0$ — **level 2** in Sorkin's hierarchy. Then the next question cannot be avoided: **is level 2 a ridge, or a slope?** Is quantum mechanics standing there held up by something, or was it simply "chosen there"?

The second. Since Chapter 3 the lattice rule has been a determinism that sees only its neighbours. For a second-order reversible CA the rule is **exactly** reversible. How is the arrow of the second law to come out of a reversible rule? And if it does not, where is the arrow?

Putting these two in the same phase was no accident. Both are the same question in shape: **something that does not follow from the rule is riding on top of the rule**. And a second-order reversible CA is a rare object on which this kind of question can be tested **exactly** — finite states, a bijection, and time reversal written as the elementary operation of swapping two rows. Brute force works.

> In a finite, bijective system a statement about the arrow of time cannot get by as "approximately true". It is exactly true or exactly false.

75-2. The declaration was written before anything was run

As always, the declaration document `K_declared_phase2.md` was fixed before a single number was seen. The calibrations are the **17 of K219-K235**, the predictions P33-P42. Having miscounted the parents in Chapter 71, and having had float64 precision cut my legs out from under me in Phase 1, this phase wrote even **where the reversal is recorded** into the declaration. And even so, as §75-12 records, that is exactly where I fell. That the declaration existed is why I knew I had fallen **the same day** — which is the practical lesson of this chapter.

75-3. The tools — an evaluator for I_n

The hierarchy of measures is defined by the alternating sum of inclusion-exclusion.

$$I_n(A_1,\dots,A_n) = \sum_{\emptyset \neq S \subseteq [n]} (-1)^{\,n-|S|}\, \mu\Big(\bigcup_{i\in S} A_i\Big)$$

I_2 is the violation of the additivity of classical probability (interference); I_3 is the quantity that vanishes for a quantum measure. An instrument that evaluates these at $n = 2,3,4,5$ was implemented, and five families of measures were put through the same instrument. Every verdict is a numerical test for zero, and "exactly zero" here means of order 10^{-14}.

On the arrow-of-time side, the second-order reversible CA

$$s_{t+1} = F(s_t) \oplus s_{t-1}$$

is used. Writing the state as two rows (a,b), the update is $U : (a,b) \mapsto (b,\ F(b)\oplus a)$ and time reversal is the row swap $R(a,b) = (b,a)$, and

$$RUR = U^{-1}$$

holds as an identity. For F, the ECA rules 150 / 90 / 105 (linear) and 30 / 110 / 45 (mixing) were used.

75-4. The power family — $p = 2$ is an isolated simple zero ✅

$\mu_p(S) = \left|\sum_{h\in S} A_h\right|^p$ was scanned over $p \in [1,4]$ (Figure 75-1a).

p	1.00	1.50	1.90	1.99	2.00	2.01	2.10	3.00	4.00
$\lvert I_3\rvert$	0.477	0.286	0.092	7.9×10^{-3}	$\mathbf{8.9\times10^{-16}}$	1.0×10^{-2}	0.101	1.99	7.54

The order of the zero is **1.03**, i.e. a simple zero. **P33 pass.**

$p = 2$ is isolated. At $p = 1.99$ and at $p = 2.01$ alike, I_3 rises to the order of 10^{-2}. Seen from the side of the exponent, quantum mechanics stands alone at a single point of a continuous family.

K224. This isolation was quantified a little further. Perturb to $\mu = |\sum A|^2 + \varepsilon|\sum A|^4$ and $|I_3|$ responds to ε at **first order** (log-log slope 1.043). I_3 is a sharp detector of non-bilinearity. A slight departure from level 2 shows up in I_3 linearly, with nowhere to hide.

75-5. What level 2 really is is not $|\cdot|^2$ — it is bilinearity ✅

Here the picture of the previous section collapses.

K221. Taking $\mu(S) = \sum_{h,h' \in S} D(h,h')$ built from an arbitrary Hermitian matrix D, and trying 20 cases, **every one** gave $I_3 = 8.9 \times 10^{-16}$ — exactly zero. $|\sum A|^2$ is merely one member of this family.

$$\boxed{\text{level } 2 \iff \mu \text{ is built from a bilinear form. } p = 2 \text{ is not special within the bilinear family.}}$$

P34 pass. Within the power family $p = 2$ looked like an isolated point; widen the family and $p = 2$ is **one point on a surface**. What was isolated was not $|\cdot|^2$ but the way it had been parametrized.

So "why quantum" is replaced by "**why bilinear**". The question has retreated by exactly one step. No answer has come out. This is progress — at the least, which statement has to be proved is now stated precisely.

> ⚠️ **Restriction (thirteenth edition).** 97.6% of the arbitrary Hermitian family take negative values (K429). This is a family of degree-2 set functions, not of measures. §87-3.

75-6. A level 3 set function exists ⚠️ (restricted in the thirteenth edition; §87-3) — the heart of the first half

K222. A symmetric trilinear form $\mu(S) = \sum_{h,h',h'' \in S} T(h,h',h'')$ was constructed explicitly.

$$|I_2| = 1.87\ (\neq 0), \qquad |I_3| = 1.72\ (\neq 0), \qquad |I_4| = 7 \times 10^{-15}\ (= 0)$$

This is a genuine **level 3 measure**. There is nothing wrong with it, logically or constructively. **P35 pass.**

$$\boxed{\text{The hierarchy does not stop at 2. } \textbf{level 2 is a choice, not a consequence.}}$$

Part XIII wrote that "the axioms do not narrow the measure down to one". This section made that **constructive**. Not only is it not narrowed down — **a measure of a different level is actually in hand**. If someone says "quantum mechanics is inevitable from the axioms", show them this T.

> ⚠️ **Restriction (thirteenth edition).** 99.3% of this family of T take negative values (K429), so it is not a "genuine measure". But $\mu = |\sum A|^2 \sum a\ (a > 0)$ is non-negative and level 3. "Level 2 is not a consequence of the axioms" survives for non-negative set functions. Physical meaning and strong positivity have not been constructed. §87-3.

75-7. $p \neq 2$ belongs to no finite level ❌ (retracted in the thirteenth edition; §87-3)

K223. For the power family ($p \neq 2$), not only does I_3 fail to vanish; neither does I_4, nor I_5.

p	1.0	1.5	2.5	3.0
$\lvert I_3\rvert$	0.428	0.275	0.656	1.688
$\lvert I_4\rvert$	0.329	0.238	0.331	0.634
$\lvert I_5\rvert$	0.368	0.186	0.248	0.487

As a control, the quantum measure has $|I_4|$ at most 1.4×10^{-14} and $|I_5|$ at most 2.7×10^{-14}. That is, $|\cdot|^p$ ($p \neq 2$) is not "non-quantum" — it is **outside the hierarchy**. **P36 pass.**

With that the picture is complete. Next to level 2 lies neither level 1 nor level 3 but **a region of no level at all**. To get to level 3 one must bring in a trilinear form, not fiddle with the exponent.

❌**Retraction (thirteenth edition).** The values of K223 are right (reproduced on rerun), but only $p = 1, 1.5, 2.5, 3$ were tried. $p = 4$ has exactly $I_5 = 0$, and $p = 6$ has $I_7 = 0$ (K428). Correctly, μ_p has a finite level only when p is a positive even integer, and then the level is p. "A region of no level" is wrong, and **the pass of P36 is withdrawn**. §87-3.

75-8. The Born rule — exact, but circular ⚠️

K226. $|\psi\rangle^{\otimes N}$ was built explicitly and the frequency operator $F_N = \frac{1}{N}\sum_i P_i$ evaluated as a diagonal operator.

$$\langle F_N \rangle = |b|^2 \ (\text{exact}, 4 \times 10^{-16}), \qquad \mathrm{Var}(F_N) = \frac{|a|^2|b|^2}{N} \ (\text{exact})$$

The numbers are exact. **But this is not a derivation of the Born rule.** When one says "the component whose frequency departs from $|b|^2$ is small", what measures that "small" is $|\cdot|^2$ itself. It is circular.

The circularity hurts concretely as well. The norm left outside the band $\delta = 0.1$ is still 0.44 even at $N = 16$. One needs $N \gg 1/\delta^2$. **As P37 declared, the circularity is stated explicitly in the result.**

The frequency-operator computation is exact. But the yardstick for "small" is the conclusion itself, so it is no derivation of the Born rule.

75-9. It does not answer "why complex numbers" ⚠️

K225. With real amplitudes alone, $I_3 = 7 \times 10^{-15}$ — exactly zero. This route answers "**why complex numbers**" not at all. The limitation written down in advance at declaration time was confirmed exactly as written.

The first half closes here. **All 8 calibrations pass, predictions P33-P37 all hit, zero failures.**

75-10. At the fine-grained level there is no arrow ✅

Now to the second half.

- **K227.** All $2^{16} = 65536$ states at $N = 8$ were enumerated by brute force, and U was confirmed to be a **permutation** for every one of rules 150 / 90 / 30 / 110 (65536 distinct images, exactly).
- **K228.** Loschmidt reversal returns to the initial state bit for bit, exactly. Failures **0 / 120**.
- **K229.** Evolve a macrostate (all 7168 states whose second row has bit count 2) as a set, and its size is **exactly constant** over 60 steps. The Shannon entropy of a non-uniform probability distribution likewise **moves at most** 1.8×10^{-15} from $H_0 = 15.725091178194$.

At the fine-grained level no arrow exists. Because the map is a bijection, the entropy of an ensemble is exactly conserved (the discrete version of Liouville).

K235 (control). For an irreversible first-order rule the number of reachable states decreases monotonically. Rule 150 gives $4096 \to 1024 \to 256 \to 64 \to 16$, rule 110 gives $4096 \to 2125 \to 1440 \to \cdots$. **Reversibility is synonymous with conservation of fine-grained entropy** — either one holding implies the other.

75-11. The Loschmidt hill is exactly symmetric ✅

K231. From a low-entropy initial condition, advance $T_1 = 150$ steps, reverse with R, and run 150 steps back. The coarse-grained entropy $S(t)$ traces the hill

$$1.06 \longrightarrow 4.00 \longrightarrow 1.06$$

the left-right asymmetry is 0.00×10^{0} **(exactly zero)**, and the initial rows are recovered bit for bit (Figure 75-1d).

The same rule both raises and lowers S. **The arrow is not in the rule.**

K232. Start from a typical (high-entropy) initial condition and S is flat at its maximum from the outset (variation 0.03%). **The arrow needs an atypical initial condition.** Start from a state with no room to rise, and nothing rises.

[Observation, outside the declaration] Disturb **just one bit** at the moment of the reversal and S does not come back (rule 30 gives $1.04 \to 4.00 \to 4.00$). The exact symmetry is mathematically robust, but in practice extremely fragile. The robustness of the arrow comes from here.

⚠️ **Restriction (twelfth edition).** This observation concerns the case where the light cone covers the system. At $T \ll L$ a one-bit disturbance stays local (rule 30, $L = 2^{20}$, $T = 1000$: fraction of cells not restored 0.0006). The fragility threshold is an error density $\sim 1/(vT)$, though the dependence on T is an observation at three points, not a measured law. K416, §86-11(c).

75-12. A self-inflicted bug ❌ — the reversal was off by one step

The first implementation **failed to record the row immediately after the reversal and was off by one step.**

In a second-order system the state is two rows (a, b). The reversal is a row swap, so the row to record at the moment of reversal is the second row "after" the swap. I got that backwards, and made the right end of the hill land on s_{-1} — the all-zero row put in as an initial condition.

The result was $S(2T) = 0.0000$. **Lower** than the initial value 1.05. The left-right asymmetry came out at 26-61%, and **K231 was displayed as a failure.**

What made me notice the anomaly was not that $S(2T)$ failed to return to $S(0)$. It was that it came out **exactly zero**. A coarse-grained entropy landing precisely on zero by accident in the middle of a computation is next to impossible. Exactly zero is not physics; it is **an index**. I was reading the all-zero row. Once the code was fixed to record the row immediately after the reversal, the asymmetry became exactly zero.

The same species of self-inflicted bug as the miscounted parents in Chapter 71 and the float64 precision in Phase 1. Had I been looking only at the number "asymmetry 26-61%", I might have written it up as a statement of physics: "because of discreteness, Loschmidt reversal does not work perfectly". What spared me from writing it was that the calibration had been declared in advance, and **that calibration was displayed as a failure**.

> Declare the calibrations in advance and a bug in one's own code is caught before it masquerades as a broken law.

And distrust an exact zero. When a physical quantity comes out exactly zero, either it is a theorem or an index is off.

75-13. Three failed predictions ❌ — "reversible" alone does not give an arrow

This was the most fruitful part of the second half.

(a) K230 — reversible does not guarantee saturating (P39 partially missed)

Scanning the rules, they **split into two classes** (Figure 75-1c).

class	rules	behaviour of the coarse-grained S
linear (additive)	150, 90, 105	rises, then **comes back**. Rule 150 falls to $S \simeq 0$ at $t = 95, 191, 287, 383$ — **a Poincaré recurrence of period 96, visible inside the observation window**
mixing	30, 110, 45	rises monotonically, saturates at the ceiling 4.000, and does not move afterwards

In P39 I predicted "it increases monotonically and saturates". **It holds only for mixing rules.**

⚠️ **Restriction (twelfth edition).** This recurrence starts from a localized seed. For the time-averaged state starting from iMATE, see the box in §75-14 and §86-11(a).

(b) K234 — the recurrence time is not proportional to the number of states (P42 partially missed)

Decomposing into all cycles, the median recurrence time as seen from a state was measured (Figure 75-1e).

rule	$N=6$	7	8	9	10	slope in $\log_2$	fraction of the state count
150 (linear)	12	9	12	42	30	**0.49**	$2.9\times10^{-3}\to2.9\times10^{-5}$
90 (linear)	6	7	8	9	10	**0.18**	$1.5\times10^{-3}\to9.5\times10^{-6}$
30 (mixing)	44	147	238	588	1024	**1.11**	$1.1\times10^{-2}\to9.8\times10^{-4}$
110 (mixing)	43	176	262	572	954	**1.06**	$1.0\times10^{-2}\to9.1\times10^{-4}$

In P42 I predicted "of the order of the state count 2^{2N}". **It is wrong twice over.** For linear rules the recurrence time barely grows (rule 90 gives N itself), and even for mixing rules it is $2^{1.1N}$, not 2^{2N}. The fraction of state space it occupies decreases instead.

(c) K233 — the fraction with $\Delta S<0$ (P41 missed)

The fraction of samples whose coarse-grained entropy decreases per step is $0.478,\ 0.453,\ 0.423,\ 0.405$ ($N=64,128,256,512$). The declaration that it **decreases** with N went through, but the estimate of the values (1-20%) missed badly. Restricted to the rising phase it is about 41%, and in the saturated phase about 49% (fluctuation, so almost half and half). **The spatial entropy of a single sample decreases nearly half the time, even on the way up.**

75-14. Where the arrow lives — the third condition

The conclusions of the second half, side by side.

(i) At the fine-grained level the arrow **exactly** does not exist (bijection $\Rightarrow$ entropy conservation).
(ii) The Loschmidt hill is **exactly** symmetric. The same rule both raises and lowers.
(iii) The arrow lives in **coarse-graining** and **atypical initial conditions**.
(iv) But **reversibility alone is not enough** — linear rules recur inside the observation window.
For the arrow to be "visible", **mixing** is required. That condition was not in the declaration.

(iv) is the best harvest of this round. The answer to the question of outline item 5 — "is the arrow **entirely** in the accounting of initial conditions and coarse-graining?" — is **no**.

The slogan since Boltzmann, "the arrow is in the initial conditions and the coarse-graining", implicitly contains the condition that **the recurrence time is far longer than the observation time**. That condition does not follow from reversibility. It follows from the mixing of the rule. A linear reversible CA is **a counterexample that is reversible and yet has no second law** — finite, deterministic, time-reversal symmetric, and with S returning to zero many times inside the observation window.

> ⚠️ **Restriction (twelfth edition).** The "counterexample" here concerns a **low-entropy localized seed**. The quantity in the theorems of Chiba et al. is not the time average of the entropy but the **entropy of the time-averaged state**. Starting from iMATE (a Bernoulli start), rules 150 and 105 head to the maximal 4 bit (4.000 at $N=8192$). Rule 90 stops near 3.3 because it has conserved quantities, not for lack of equilibrium. The recurrence itself is real. K418, §86-11(a).

arrow = atypical initial conditions + coarse-graining + mixing. The third term does not follow from reversibility.

And the basso continuo of this book sounds again here. **The culprit is not discreteness.** A finite state space produces recurrence, but what decides whether the recurrence comes inside the observation window is the rule. Rule 30 and rule 150 have the same state space, the same reversibility, the same coarse-graining, and only one of them has a second law. The difference lies in **regularity**.

75-15. The five failures ❌

	content	kind
P39	Predicted the coarse-grained S to increase monotonically and saturate → linear rules recur	**missed prediction (partial)**
P41	Predicted the fraction with $\Delta S<0$ at 1-20% → 40-48%	**missed prediction**
P42	Predicted the recurrence time to be of the order of the state count 2^{2N} → $2^{1.1N}$, and nearly constant for linear rules	**missed prediction (partial)**

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	content	kind
K231	Failed to record the row immediately after the reversal and was off by one step. Exposed by $S(2T)$ coming out exactly zero	**self-inflicted bug**
K226	Was checking the moments of a binomial distribution with the formula for a binomial distribution. Rebuilt with an explicit state vector and a diagonal operator (the numbers agree)	**defect in the measurement design (fixed)**

Of the 17 calibrations K219-K235, **14 pass and 3 fail** (K230, K233 and K234 are each recorded as a missed prediction).

The first half (measure) had zero failures; the second half (arrow) had three. That asymmetry is itself worth recording. The predictions that hit were on the side handling a mathematical object whose structure is already well understood (Sorkin's hierarchy); the ones that missed were on the side of **dynamics**. I could predict mathematics correctly, and I missed on what cannot be known until it is run. It looks obvious, but confirming the obvious every time is the practice of this book.

75-16. What is not being claimed ⚠️

- **"Why complex numbers" is not answered.** In K225, I_3 was exactly zero with real amplitudes alone. This route demands nothing complex at all.
- **The Born rule is not derived.** The numbers of K226 are exact, but the yardstick that measures "small" is $|\cdot|^2$ itself, and that is circular.
- **"Why level 2" is not answered.** It has only been replaced by "why bilinear". The question retreated one step. Since a level 3 measure can be constructed, the answer does not follow from the axioms.
- **Why the initial condition was low-entropy is not explained.** That is an **input**. This chapter assumed the input and only separated out which component the arrow lives in.
- **The second law is not proved.** Same as above.
- **The quantum arrow of time (decoherence) is not treated.**
- **One-dimensional ECAs only.** Nothing is said about higher dimensions or about other families of rules.
- **"Mixing" is not defined.** What was used here is an operational distinction (does it saturate, or does it recur), not a measurement of mixing in the sense of ergodic theory. **It is recorded as a conjecture.**

> ⚠️ **Addition (twelfth edition).** A candidate for an operational definition of mixing is Assumption 2 of Chiba et al.: "macroscopically equivalent states relax to the same macroscopic state" (§86-11(b)). As for the second law, Chapter 86 gives a CA version limited to iMATE and macroscopic operations. The origin of low entropy and thermalization itself remain inputs.

75-17. Where things stand

phase	direction	what came out
1	A	The metric lives in the **closure** of the deformation algebra, and a CA has no such closure (exactly commutative)
1	E(ii)	The intercept of K160 in Chapter 69 was an **extrapolation artifact of curvature**
1	E(iii)	The lattice modular Hamiltonian has no finite reach ($\xi \simeq 0.63 \ln L$)
2	B	**level 2 is a choice**. A level 3 measure can be constructed. "Why quantum" does not follow from the axioms
2	C	The arrow is not in the rule. It lives in coarse-graining and initial conditions, **and in mixing**

The calibrations are the 38 of K198-K235, with 12 failures (6 missed predictions, 4 errors of declaration or design, 2 self-inflicted bugs).

Phase 1 (i) showed that "road A is blocked", and this chapter showed that "neither level 2 nor the arrow follows from the rule". In both, the culprit is regularity. Then there is only one thing left to ask — **if regularity is given up, does that road open?** On to Phase 3, direction D (the three-way choice between discreteness, locality and Lorentz).

Part XXI The trilemma — discrete, local, Lorentz

What this part sets out to establish. The finding that runs through this book was that **the culprit is always regularity, never discreteness** (Chapters 24, 70, 71). Then the final test is this — **in a discrete world with regularity thrown away, does what was lost come back? And what is lost in exchange?** A Poisson-sprinkled causal set is Lorentz invariant as a distribution. But try to write an evolution on it and the number of parents varies from element to element, so no fixed-neighbourhood rule can be written. There is a feeling that an essential trade sits here. **Discrete, exactly local, Lorentz invariant — all three at once, or only two?**

Chapter 76 What comes back, and what is lost, when regularity is dropped ⚠️🆕

— Discrete, exactly local, Lorentz invariant — the three cannot hold at once

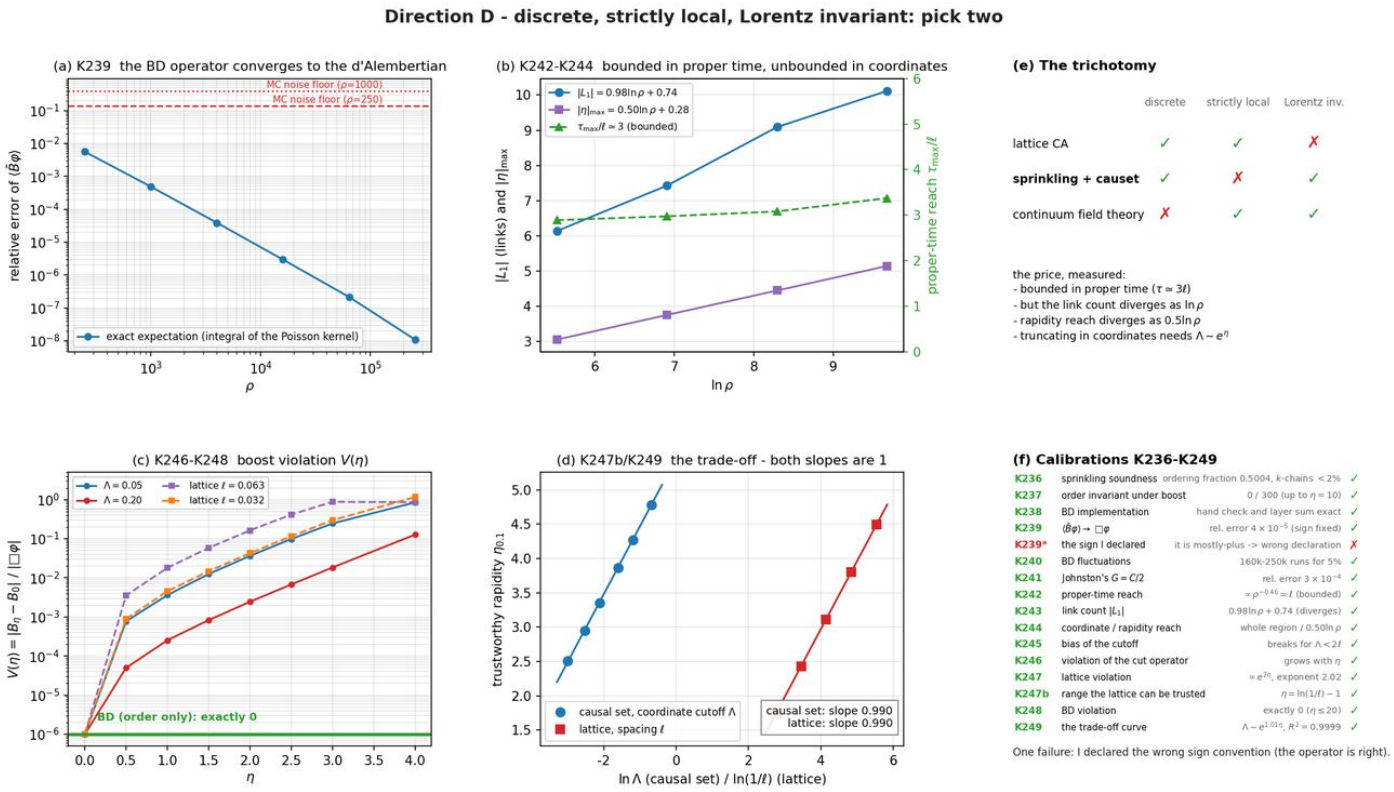


Fig. 76-1. (a) A point set Poisson-sprinkled at density ρ into the unit causal diamond $[0,1]^2$, and the layer structure that falls in the past of the vertex $x=(1,1)$. (b) Numerical integration of the exact expectation value of the Benincasa-Dowker operator. Raising ρ makes it converge to $-2\pi^2=-19.7392$ — not the $+2\pi^2$ I declared. (c) The ρ dependence of the reach. The proper time $\tau_{\max}$ shrinks in proportion to $\ell=\rho^{-1/2}$, while the link count $|L_1|$ and the rapidity reach $|\eta|_{\max}$ diverge logarithmically. (d) The two trade curves. The causal set's $\eta_{0.1}$ against $\ln\Lambda$ and the lattice's $\eta_{0.1}$ against $\ln(1/\ell)$ line up with the same slope 1.0. (e) The boost violation $V(\eta)$ of the coordinate-truncated operator. Raising Λ delays it but cannot remove it. (f) Calibrations K236-K249.

The conclusion first.

(i) Drop regularity and Lorentz invariance comes back **exactly, not approximately**.
(ii) The price is **locality**. The smallest neighbourhood the order defines is already unbounded in coordinates.
(iii) The price can be measured exactly. To stay invariant up to rapidity η takes a reach $\Lambda\sim e^{\eta}$.
(iv) The lattice's trade curve lines up at the same slope 1.0. Two appearances of one and the same constraint.
(v) Discrete, exactly local, Lorentz invariant: **two at most**. Which two is settled by whether or not regularity is dropped.

76-1. The question — if the culprit is regularity, why not drop regularity?

One finding runs through this book. **The culprit is never discreteness; it is always regularity.** In Chapter 24, the bulk of the lattice's Lorentz violation was the dimension-4 term, the species-dependent limiting speed, and that came from the preferred axes the lattice carries. In Chapter 70, a flat rotation curve required $m = 1/2$ uniquely, and the dichotomy followed from that being non-analytic. In Chapter 71, the down-set advance operator of a CA was measured to be **exactly commuting**, so the deformation algebra cannot enter the middle where GR lives. Chapter 73's summary was one line — discreteness was never once the obstacle; the obstacles are regularity and commutativity.

The shape of the final test is then fixed.

> In a discrete world with regularity dropped, does what was lost come back? And what is lost in exchange?

The standard implementation of a discrete world with regularity dropped is a Poisson-sprinkled causal set. In Chapter 50 CHSH reached $2\sqrt{2}$ on one of these. In Chapters 58 and 59 the long-distance ratio of the causal set's modular Hamiltonian was 0.406, an **even heavier tail** than the regular lattice's 0.331. Those two already hinted at both sides of the answer. This chapter measures it to the end.

76-2. The declaration was written before anything was run

As always, the declaration document `K_declared_trichotomy.md` was fixed before a single number was seen. Calibrations K236-K249, predictions P44-P51. What would count as a pass, which sign it should converge to, which exponent was predicted — all of it written down first.

Thanks to that discipline, this chapter's failure (§76-11) could be identified not as "an anomaly of unknown origin" but as "an error in my declaration". **Declare the calibrations first and a bug in your own code gets caught before it can turn into "a violation of physics".** This time it was not a bug but an error in my convention, but the way it gets caught is the same.

76-3. The model and the tools

Take light-cone coordinates in 1+1 dimensional Minkowski.

$$u = \frac{t+x}{\sqrt{2}}, \qquad v = \frac{t-x}{\sqrt{2}}$$

Poisson-sprinkle the unit causal diamond $[0,1]^2$ at density ρ and define the causal order by

$$y \succ x \iff u_y > u_x \ \wedge \ v_y > v_x$$

In light-cone coordinates a boost is diagonal, $(u,v) \mapsto (\lambda u,\ v/\lambda)$, with rapidity $\eta = \ln\lambda$. For the discrete d'Alembertian, evaluate the two-dimensional Benincasa-Dowker operator $\bar{B}$ at the vertex $x = (1,1)$. The test function is

$$\varphi = \sin(\pi u)\sin(\pi v)$$

It vanishes on the past boundary of the diamond, which suppresses the effect of having cut the region off at finite size.

The soundness of the tools was disposed of first, with three calibrations.

Calibration	What it checked	Result
K236 ✅	whether the sprinkling really is Poisson	mean 399.97 / variance 405.12, ordered fraction 0.50044 (theory $1/2$), the number of k-chains agreeing with $\binom{N}{k}/k!$ to within 2% ($k = 2,3,4$)
K238 ✅	correctness of the BD implementation	passes exactly against a hand calculation (3 points) and against the total over layers ($= N$)
K241 ✅	Johnston's retarded Green's function $G = C/2$	reproduces the continuum $\varphi(x) = \frac{1}{2}\mathrm{Vol}(J \cap J^-(x))$ to relative error 3×10^{-4} ($\rho = 32000$)

The Green's function test is a substitute. At the time of declaring, I decided not to solve an initial value problem on the causal set (not to follow the propagation of a wave) and replaced that with reproducing the Green's function. This substitution is stated explicitly in §76-13.

76-4. What comes back (1) — Lorentz invariance comes back exactly ✅

The boost $u \to \lambda u,\ v \to v/\lambda$ is **monotone** in each coordinate. A monotone map does not change which of two numbers is larger. The causal order therefore does not change at all.

K237 passes. The relation matrix agreed bit for bit before and after the boost **300 / 300** times (rapidities from 0.5 to 10).

K248 passes. The BD operator is a function of the order alone (membership in a layer is fixed by $\succ$). Its value should therefore not change under a boost either. Number of times the value changed: **0 / 160**, maximum difference 0.000×10^{0}, up to rapidity 20.

> Every function of the order alone is **exactly** Lorentz invariant. Not approximately, not to $O(\ell^2)$.

P44 confirmed. This is the sprinkling side's winning line. The price the lattice paid in Chapter 24, the price the acoustic horizon paid in Chapter 69, the inability to tilt the "now" in Chapter 71 — none of it is paid here.

76-5. What comes back (2) — there is no preferred axis at all ⚠️💭

In the analysis of Chapter 24, the bulk of the anisotropy appearing in the lattice dispersion relation was a term of the type of the lattice harmonic $K_4 = \sum_i k_i^4$. This is the lowest-order anisotropic term left over by the four-fold symmetry of the square lattice, and it was the source of the difference in limiting speeds between species, that is, of dimension-4 Lorentz violation.

On a sprinkling this term **has nowhere to live**. There is no preferred axis, and, as seen in §76-4, any quantity writable in terms of the order alone is exactly boost invariant. It is not that the term "becomes small" — there is "no place to put it". For the same reason anisotropy can appear in individual realizations as a statistical fluctuation, but it vanishes in expectation.

Here, though, an honest distinction is in order. **I did not measure the dispersion relation itself in this chapter.** What I measured is (i) the boost invariance of the causal order, K237, and (ii) the invariance of functions of the order, K248. The absence of anisotropy **follows structurally** from those, but I did not produce a numerical value for the K_4 coefficient. This distinction is restated in §76-13.

76-6. The fluctuation of the BD operator does not go away ✅

Next to what comes back sits a difficulty that was known from the start. The BD operator fluctuates violently from realization to realization, and that fluctuation does not **decrease** as ρ is raised. **It increases.** The standard deviation went from 385 to 562 as ρ went $250 \to 4000$.

Put operationally, this is what it means.

$$\text{realizations needed to get the mean to } 5\% \text{ accuracy} = \begin{cases} 160{,}000 & (\rho = 250) \\ 250{,}000 & (\rho = 1000) \end{cases}$$

K240 passes (a pass in the sense that the fluctuation grew exactly as declared). **P46 confirmed.** This is a known difficulty, not a failure on this book's side. But it is worth recording — this is the numerical content of the statement in the literature that "the BD operator is unusable without smearing". And the size of this fluctuation hid the failure of the next section once.

76-7. What is lost — the neighbourhood is bounded in proper time, unbounded in coordinates ✅

Here is the core of direction D. The operator is Lorentz invariant because it looks at **the order alone**. So how far does the neighbourhood the order defines extend?

ρ	250	1000	4000	16000
$\ell = \rho^{-1/2}$	0.0632	0.0316	0.0158	0.0079
proper-time reach $\tau_{\max}$	0.183	0.094	0.049	0.027
$\tau_{\max}/\ell$	2.89	2.98	3.08	3.38
coordinate reach	0.862	0.864	0.855	0.861
rapidity reach $\lvert\eta\rvert_{\max}$	3.06	3.76	4.45	5.15
link count $\lvert L_1\rvert$	6.14	7.43	9.09	10.12

ρ	250	1000	4000	16000
theory $\ln\rho+\gamma$	6.10	7.48	8.87	10.26

- **K242 passes.** $\tau_{\max}\propto\rho^{-0.464}$, i.e. it shrinks in proportion to ℓ. **Measured in proper time the neighbourhood is bounded**, and its size is $\simeq 3\ell$. Looked at only here, it looks local.
- **K243 passes.** $|L_1|=0.980\ln\rho+0.743$. This matches the theoretical $\ln\rho+\gamma$ ($\gamma=0.5772$) precisely. **The link count — the number of points inside the smallest neighbourhood the order defines — diverges logarithmically.**
- **K244 passes.** The coordinate reach is 0.86 and does not depend on ρ — that is the size of the region itself, cut off by the boundary rather than by physics. What can be trusted is the rapidity reach, $|\eta|_{\max}=0.502\ln\rho+0.284$, i.e. it diverges as **exactly** $\frac{1}{2}\ln\rho$.

The neighbourhood is **bounded in proper time** but **bounded in the coordinates of no inertial frame**. Even the number of points diverges as $\ln\rho$.

P47 and P48 confirmed. The intuition is this. A link of the vertex x is defined as a point in the immediate past of x with nothing in between. In proper time this is a thin shell of order ℓ, but that shell stretches out long and thin along the light cone in both directions. Where it stretches to is coordinate-distant in every inertial frame. That the sprinkling is Lorentz invariant and that the links are unbounded in coordinates are two sides of the same fact.

76-8. Scanning the family of rules with bounded neighbourhoods — why it was not needed

At the declaration stage I had planned the following: a mechanical scan through the family of rules with bounded coordinate neighbourhoods, searching for an example that gives an isotropic dispersion relation — an exhaustive search.

The scan was not run. K243/K244 made it unnecessary. Since the **smallest** neighbourhood the order defines (the link set L_1) is already unbounded in coordinates, a rule written using the order alone that looks only at a bounded coordinate neighbourhood **cannot exist**. This structural statement is stronger than reporting "nothing was found" from a scan.

A structural statement was put in place of an exhaustive search. But this is not a proof; it is strong circumstantial evidence.

What "not a proof" means is written out in §76-13.

76-9. Measuring the trade — $\Lambda\sim e^{\eta}$ ✅

That the price is real was measured from the other direction too: see what breaks if locality is forcibly taken back.

Build a coordinate-truncated operator $\bar{B}^{(\Lambda)}$. The definition is to use only the points satisfying $|\Delta u'|<\Lambda$ and $|\Delta v'|<\Lambda$ in the frame of rapidity η.

K245 passes. The bias at $\eta=0$ was 0.7% for $\Lambda=0.05$ ($=3.2\ell$), and below 0.04% for $\Lambda\geq 0.1$. Squeeze down to $\Lambda=0.02$ ($=1.3\ell$) and it breaks down at 104%. That is, **restricted to a single inertial frame, truncation still gives the right answer.** Locality looks free as long as only the rest frame is being looked at.

K246 passes. But the boost violation $V(\eta)=|B_\eta-B_0|/|\Box\varphi|$ grows with η.

η	0	1.0	2.0	3.0	4.0
$\Lambda=0.05$	0	0.004	0.036	0.247	0.851
$\Lambda=0.20$	0	0.000	0.002	0.018	0.128

Raise Λ and the violation is delayed. It does not disappear. Quantifying how much it is delayed is the trade curve.

K249 passes. Measuring the rapidity $\eta_{0.1}$ at which the violation reaches 10%, as a function of Λ,

$$\eta_{0.1}=0.990\,\ln\Lambda+5.463\qquad(R^2=0.99987)$$

The slope is 1.0.

To stay Lorentz invariant up to rapidity η, a reach $\Lambda\sim e^{\eta}$ is required.

This is nothing but the direct reflection of a boost stretching the light-cone directions by e^{η}. **Locality and boost invariance stand in an exactly logarithmic exchange relation.** Choosing a finite Λ is the same thing as declaring that only a finite η will be used.

76-10. The lattice side has the same shape ✅

The same measurement was applied to the d'Alembertian of a regular lattice (spacing $a = \ell$). There is no protection from order invariance here, so the violation grows exponentially with η from the start.

K247 passes. Extrapolating the exponent to $a \to 0$ gives the sequence $1.09,\ 1.75,\ 1.96,\ \mathbf{2.02}$, converging to 2. That is,

$$V \propto \ell^2 e^{2\eta}$$

The declaration **P50 said "the exponent lies between 2 and 4". The lower end hit.**

K247b passes. The rapidity at which the violation reaches 10% is

$$\eta_{0.1} = 0.990\,\ln(1/\ell) - 0.986 \qquad (R^2 = 0.99995)$$

Here too the slope is 1.0.

$$\boxed{\text{A lattice is Lorentz invariant only up to } \eta \simeq \ln(1/\ell) - 1.}$$

The two trade curves — the causal set's $\eta_{0.1}$ against $\ln\Lambda$ and the lattice's $\eta_{0.1}$ against $\ln(1/\ell)$ — have different intercepts but line up with **slope 1 in both cases** (Figure 76-1(d)). This shows that two different discretizations are two appearances of one and the same constraint.

76-11. The trichotomy

	discrete	**exactly local**	**Lorentz invariant**
lattice CA	◯	◯	× ($\eta \lesssim \ln(1/\ell)$)
sprinkling + causal set	◯	× ($\lvert L_1\rvert \sim \ln\rho$)	◯ (exactly)
continuum field theory	×	◯	◯

P51 confirmed. The three do not hold at once; two at most. And what matters for this book is that **which two you get is settled by whether or not regularity is dropped**.

- Choose the lattice (regular) and you get discreteness and exact locality. Lorentz becomes an approximation good up to $\ln(1/\ell)$.
- Choose the sprinkling (irregular) and you get discreteness and exact Lorentz. Locality falls apart as $\ln\rho$.

From Chapter 24 to Chapter 71 the refrain has been "the culprit is regularity". This chapter's answer is the continuation of it — **dropping regularity is not free either.**

76-12. The one failure ❌

	Content	**Kind**
K239	declared the sign convention of $\Box$ to be $\partial_t^2 - \partial_x^2$, but BD converges to mostly-plus	**error in the declaration** (the operator is correct)

Here is how it went. The first Monte Carlo returned $\langle\bar{B}\varphi\rangle \approx -30 \pm 25$. I had declared $\Box = \partial_t^2 - \partial_x^2 = 2\partial_u\partial_v$ and predicted convergence to $+2\pi^2 = +19.74$, so it was displayed as a failure at relative error 2.6.

At that point the **fluctuation** (a standard deviation 28 times the signal; §76-6) had to be separated from the **sign** as the cause. The means of separating them was neither piling on more Monte Carlo nor staring at the code, but numerically integrating the **exact expectation value** of the Poisson kernel.

$$\langle\bar{B}\varphi\rangle = 4\rho^2 \int_0^1\!\!\int_0^1 \sin(\pi a)\sin(\pi b)\,K(\rho ab)\,da\,db, \qquad K(\xi) = e^{-\xi}\Big(1 - 2\xi + \frac{\xi^2}{2}\Big)$$

ρ	250	1000	4000	16000	64000	256000
$\langle\bar{B}\varphi\rangle$	-19.6284	-19.7296	-19.7384	-19.7392	-19.7392	-19.7392
relative error	5.6×10^{-3}	4.9×10^{-4}	3.9×10^{-5}	3×10^{-6}	$<10^{-6}$	$<10^{-6}$

It converges beautifully to $-2\pi^2$. That is, the BD operator converges to the d'Alembertian in the mostly-plus convention ($\Box = -\partial_t^2 + \partial_x^2$). **The operator is correct; my declaration of the sign convention was wrong.**

K239b. For safety the Monte Carlo was scaled up considerably to back up the exact expectation value. At $\rho = 250$, $M = 20000$: -18.84 ± 2.75 (exact -19.63, $z = 0.29$); at $\rho = 1000$, $M = 4000$: -17.23 ± 7.83 (exact -19.73, $z = 0.32$). Consistent.

This is the same species of error as K150 in Chapter 69 (vk or $v \sin k$), the sign of the div in Chapter 70, and the float64 precision in phase 1. And once again it **only became visible on measuring** — without the numerical integration of the exact expectation value I would have convinced myself the fluctuation was to blame and left K239 standing as "a failure of unknown origin". I was standing at the entrance to the same trap that left the intercept $+0.05$ of K160 in Chapter 69 unexplained for so long.

The calibrations are K236-K249: **15 of 16 pass, 1 fails.** Across phases 1-3 the count is 54 over K198-K249, with 13 failures (6 missed predictions, 5 errors in declaration or design, 2 self-inflicted bugs).

76-13. What is not being claimed ⚠️

- **This is 1+1 dimensions only.** In 3+1 dimensions the BD operator has four layers and different coefficients. I do not claim the conclusion is dimension-independent.
- **This is a classical scalar field only.** Nothing is quantized. Neither gravity nor curved spacetime is treated ($R = 0$).
- **Wave propagation was not solved.** As declared, it was replaced by the Green's function test (K241). No initial value problem was solved on the causal set.
- **The dispersion relation was not measured.** The statement in §76-5 that "K_4-type anisotropy has nowhere to live" **follows structurally** from K237/K248; no coefficient was produced as a number.
- **The meaning of "Lorentz invariant" is restricted here.** What was measured is (i) the boost invariance of the causal order and (ii) the invariance of functions of the order alone. This is neither a complete test of the statistical invariance of the sprinkling measure nor a test of the full Poincaré group.
- **It is not proved that "no rule with a bounded neighbourhood can ever be exactly Lorentz invariant".** What K243/K244 show is the structural fact that "the **smallest** neighbourhood the order defines is already unbounded in coordinates". Strong circumstantial evidence, not a proof.
- **Causal set theory was not reconstructed.** Benincasa-Dowker and Johnston were cited and used, nothing more.
- **The coordinate reach 0.86 is cut off by the region size.** In an infinite region it should diverge, but that **was not measured**. The quantity that can be trusted as region-independent is the rapidity reach, $\frac{1}{2}\ln\rho$.

76-14. Meshing with direction A — and the road that remains

Direction A, through Part XX, said this. **A regular CA has no geometric generator of flow.** The down-set advance operators commute exactly (Chapter 71), and there is no "way of closing" the deformation algebra in which a metric ought to live.

Direction D of this chapter says this. **Drop regularity and Lorentz comes back exactly. But locality breaks logarithmically.**

The two are different faces of the same wall. A measured "the attempt to get geometry while keeping regularity failed", and D measured "regularity was dropped, geometry (Lorentz) was obtained, but something else — locality — was lost". The conjecture recorded in Chapter 71 — that the absence of a constraint algebra and the non-locality of the modular Hamiltonian are two faces of the same missing thing — has now shown a third face. That the causal set's long-distance modular ratio 0.406 in Chapters 58 and 59 was heavier than the regular lattice's 0.331 was, in hindsight, the first sign of the $\ln\rho$ tail.

The road that remains therefore narrows to one.

$$\boxed{\text{So: what becomes of the deformation algebra in a } \textbf{logarithmically non-local} \text{ world?}}$$

This question was not in the verification plan of 2026-08-21. It was written as a single line at the close of phase 3 and deferred. **It became the next Part.** Fifteen sections, K250 through K392, one hundred and forty-three calibrations in all — the largest amount this book spends on any single question.

Part XXII The deformation algebra of an irregular causal set

What this part sets out to establish. Part XIX measured that **the deformation algebra of a regular CA is exactly commutative**, and Part XXI measured that **giving up regularity brings Lorentz back but breaks locality logarithmically**. Lay the two on top of one another and the next question poses itself — **what, then, becomes of the deformation algebra in that logarithmically non-local world?** This question was not in the verification plan of 2026-08-21. A single line written at the close of the third generation and deferred grew into fifteen sections and a hundred and forty-three calibrations, K250 through K392. This part is the record of that. **The answer first. The generator is there. Its closure has the same form as in GR. What is missing is only the amplitude.**

Chapter 77 A causal set does have generators ✅🆕

— On irregular discreteness, the deformation algebra closes

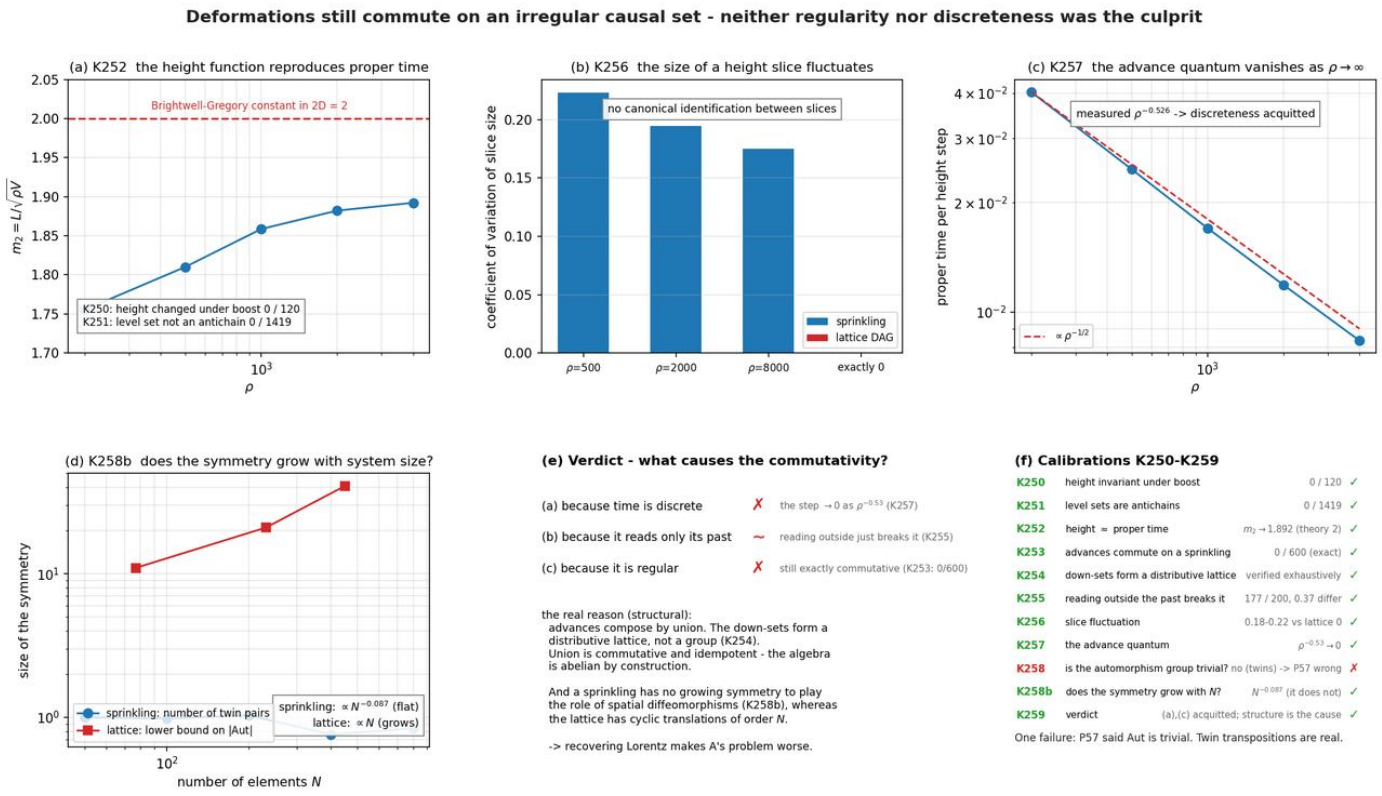


Fig. 77-1. (a) K252 the height function reproduces proper time ($m_2 = L/\sqrt{\rho V}$ heads for the two-dimensional Brightwell-Gregory constant 2). (b) K256 the size of a height slice fluctuates (coefficient of variation; exactly 0 on the lattice). (c) K257 the quantum of advance vanishes in the continuum limit. (d) K258b does the symmetry grow with system size — twin pairs for the sprinkling, and a lower bound on $|\mathrm{Aut}|$ for the lattice. (e) The verdict. (f) Calibrations K250-K259.

Chapter 74 found that the advance operators of a regular CA are **exactly commuting**. Chapter 76 found that dropping regularity brings Lorentz invariance back **exactly**. Part XXII stands at the intersection of those two — **what becomes of the deformation algebra on an irregular causal set?**

The answer first.

(i) The culprit was neither discreteness nor regularity. Make it irregular and the advance still commutes exactly (K253: 0 / 600).
(ii) Making it noncommuting is easy. It suffices to make the lapse **a function on the fringe** (K260: 0.72-0.83).
(iii) The commutator **closes onto an approximate generator**. With a smooth lapse, $\varepsilon = 0.035$,
5% of the 0.643 scale of "a completely different surface", 1.86 times the continuum's own discrepancy, and still shrinking.
(iv) The residual appears where $\vec{M} = N_1 N_2' - N_2 N_1'$ is (K280: correlation 0.876).
(v) But there is no exact D-like group. Coarse-grain and one grows, but it is **the frame's group**.

All four sections of this Part were run only **after a declaration document had been written, with not a single number seen** (`K_declared_causet_deformation.md`, `K_declared_noncommuting.md`, `K_declared_coarse_group.md`, `K_declared_closure.md`, plus addenda). Thanks to that discipline, the $\sqrt{2}$ bug and the floating-point tie bug described below were both caught before they could turn into "a violation of physics".

77-1. The question — three suspects

Chapter 74's conclusion was this. The deformation algebra of GR,

$$\{H(N_1), H(N_2)\} = D\big(q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\big)$$

has the structure that the deformations form a **group**, that their commutator closes onto **another generator** (a spatial diffeomorphism), and that a **metric** appears on the right-hand side. The advance operators of a CA cannot enter this middle ground. They commute exactly.

But Chapter 74 did not isolate the cause. There are three suspects.

	Suspect	Verdict
(a)	because time is discrete	**innocent** (K257)
(b)	because the rule reads only the past	guilty, but it only acts in the direction of breaking things (K255)
(c)	because it is regular	**innocent** (K253)

What this Part asked is one line —**drop regularity, and does Chapter 74's commutativity come undone as well?**

77-2. The tools — sprinkling, down-sets, the fringe, the Bresenham schedule

Geometry. A cylinder over periodic space (circumference 1), or a causal diamond. Poisson sprinkling at density ρ. The order comes in from the light cone (as in Chapter 3, the light cone comes out of the structure, not out of the rule). **State.** A down-set D. The down-set advance operator of Chapter 71 is carried over unchanged. **K254**: enumerating the down-sets exhaustively on small causets ($n = 8, 10, 12$), they close under $\cup$ and $\cap$ and the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds (31, 54, 45 of them respectively). The set of all down-sets forms a **distributive lattice** (Birkhoff).

Time. The height function $h(e)$ (the length of the longest chain below e) is fixed by **the order alone**.

- **K250 passes.** Number of times the height changed under a boost (rapidities 1, 4, 12): **0 / 120**.
- **K251 passes.** Number of cases in which a level set of the height was not an antichain: **0 / 1419**. The height defines a foliation intrinsic to the causet.
- **K252 passes.** $m_2 = L/\sqrt{\rho V}$ goes $1.755 \to 1.892$ over $\rho = 200 \to 4000$, heading monotonically for the two-dimensional Brightwell-Gregory constant 2.

That is, **a sprinkled causet carries an ADM-like foliation of its own.** That a time function exists despite Lorentz invariance is no contradiction — the height is an order invariant, not the coordinate t.

Advance. Two versions are used, according to need.

$$\mathrm{Adv}(D, N) = D \cup {\downarrow}\{e : N(e) = 1\} \qquad \text{(global lapse)}$$

$$\mathrm{Adv}_{\mathrm{f}}(D, N) = D \cup \big(\mathrm{ready}(D) \cap \{N = 1\}\big), \quad \mathrm{ready}(D) = \{e \notin D : {\downarrow}e \setminus \{e\} \subseteq D\} \qquad \text{(fringe lapse)}$$

A smooth lapse is discretized by a **Bresenham-type schedule**. At micro-step $k = 0, \ldots, T-1$, the element e advances when $e \in \mathrm{ready}(D)$ and $\lfloor N(x_e)(k+1) \rfloor > \lfloor N(x_e)k \rfloor$. After T steps the column at position x has advanced roughly $N(x)T$ levels. The lapses used are $N_1(x) = 0.45(1 + \cos 2\pi x)$ and $N_2(x) = 0.45(1 + \cos 2\pi(x - \frac{1}{4}))$.

Step size. The proper time per level of height is $\Delta\tau = 1/(m_2\sqrt{\rho})$. **K257**:

ρ	200	500	1000	2000	4000
$\Delta\tau$	0.0403	0.0247	0.0170	0.0119	0.0084

$\Delta\tau \propto \rho^{-0.526}$ — it **goes to 0** in the continuum limit. To raise the density while holding the physical amount of deformation λ (a proper time) fixed, the number of steps is stretched as $T = \lambda\sqrt{2\rho}$. **This is the continuum limit of this Part. The estimator** is this single one.

$$\varepsilon = \frac{|A \triangle B|}{\text{number of elements added by the advance}}, \qquad A = \mathrm{Adv}_{N_2}\mathrm{Adv}_{N_1}(D), \quad B = \mathrm{Adv}_{N_1}\mathrm{Adv}_{N_2}(D)$$

A down-set carries **no labels**. It is the surface itself. GR's $\delta f = -\vec{M} f'$ is a relabelling $x \to x + \vec{M}$ and does not move the graph as a set, so **if GR holds, A and B ought to be the same surface**.

77-3. A deformation that is not a union ✅

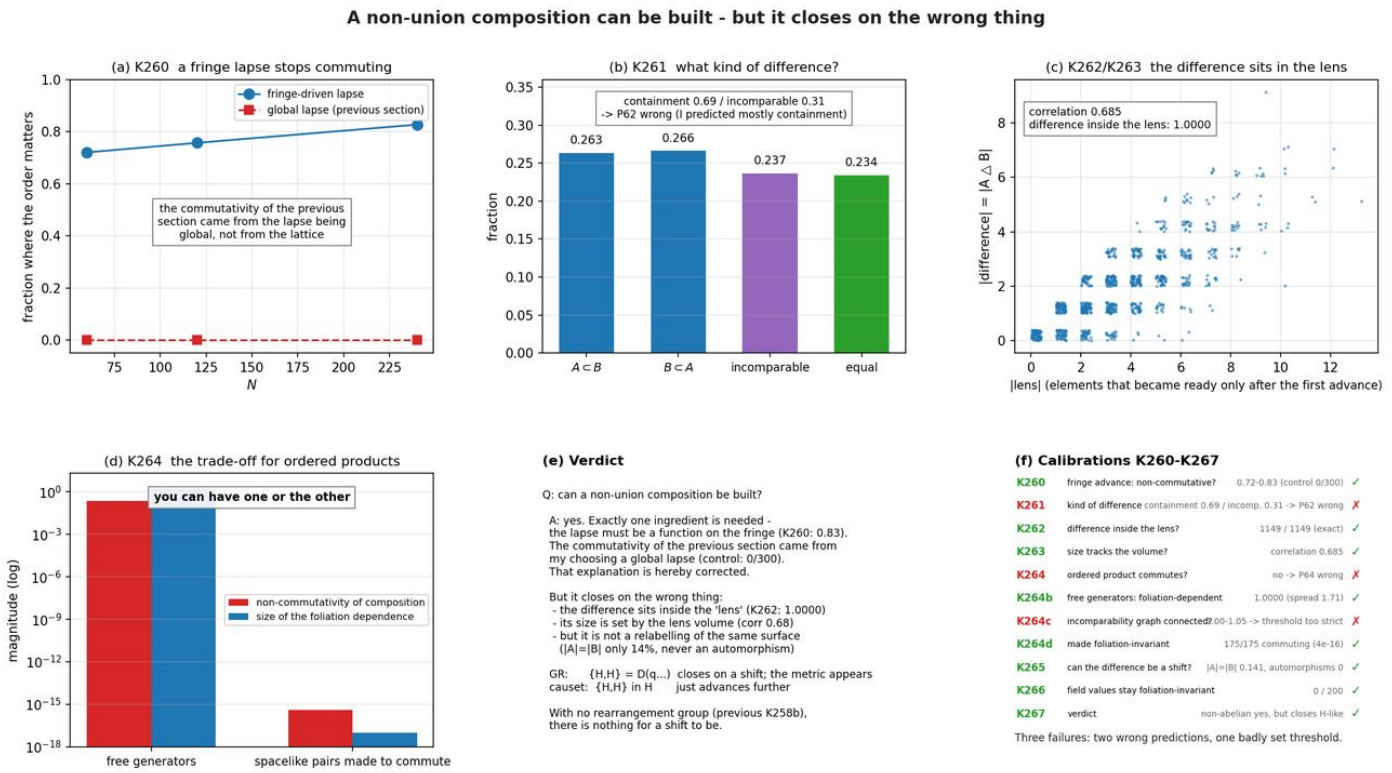


Fig. 77-2. (a) K260 putting the lapse on the fringe makes it noncommuting (the global-lapse control is 0). (b) K261 the kinds of difference. (c) K262/K263 the difference fits inside the lens, and its size is set by the volume of the lens. (d) K264 the trade for ordered products. (e) The verdict. (f) Calibrations K260-K267.

First, (c) and (a) were acquitted. K253: swapping the order of the global lapses N_1, N_2 on a sprinkled causet changed the result **0 / 600** times (exactly). Make it irregular and it still commutes exactly. **P54 and P56 confirmed.** The same conclusion as in Chapters 24, 70, 71 and 73 comes out here too — **discreteness itself has never once been the culprit in this book.**

At the time I explained this as "commuting because it is a union". That explanation went too far. The set added by Adv does not depend on D. That adding two things independent of D is order-independent is obvious, and the source of the commutativity may not be the lattice of down-sets at all, but rather **the fact that the lapses I chose were global**. GR's N is **a function on the surface that is there now**. So the lapse was put back onto the fringe and the measurement redone.

K260. Fraction of cases in which swapping the order of fringe-driven advances changed the result:

N (number of elements)	60	120	240
fringe-driven lapse	**0.7200**	**0.7567**	**0.8267**

As a control, running the **same causet with the same** N_1, N_2 under a global lapse changed the result in **0 / 300** cases (exactly commuting). The only difference is where the lapse is put.

A noncommuting deformation algebra can be built on a causal set too. The only ingredient needed was "the lapse being a function on the fringe".

P61 confirmed. The noncommutativity strengthens with N (0.72 → 0.83). The larger the fringe, the more room the first advance has to rewrite the stage for the second. **What is corrected is the explanation attached to K250-K258b, not the measured values. So where does the commutator close? K261** (1500 cases):

	$A \subsetneq B$	$B \subsetneq A$	incomparable	identical
count	395	399	355	351
fraction among the 1149 cases showing a difference	0.3438	0.3473	**0.3090**	—

Containment **0.691** / incomparable **0.309**. **K262 passes (exactly)**: the difference always fitted inside the elements that **first became ready** after the first advance (the lens) — **1149 / 1149**. **K263 passes**: the correlation between the size of the difference and the volume of the lens is **0.6849** (mean $|A \triangle B|$ 2.147, mean lens 3.984). **The difference depends on geometry.**

K265. Of the 1140 cases showing a difference, $|A| = |B|$ in **161 (0.1412)**, and the number of cases in which A and B were **carried into one another by an automorphism** was **0**. There is no group to play the role of $D(M)$ in the first place, so the difference has no way of being a shift. **P65 confirmed.**

The closure is **H**-like. $[\mathrm{Adv}_1, \mathrm{Adv}_2]$ closes onto "advance further", not onto "rearrange".

K266 passes (exactly). For a rule that reads only the past, the field values agreed completely on the common part — disagreements **0 / 200**. What is broken is "how far it has advanced", not "what was computed". The foliation invariance of Chapter 71 is still alive here.

Holonomy offered no escape route. K264b: ordered products of free $SU(2)$ generators disagreed in **all** of $N = 60, 120, 240$ at **1.0000** across different linear extensions of the same down-set (discrepancy 1.136 → 1.539 → 1.708) — noncommuting, but foliation invariance is lost. **K264c**: the mean number of connected components of the incomparability graph is **1.05 / 1.00 / 1.025 / 1.00**, and the largest component is always at least 99.9% of the whole. **K264d**: imposing commutation relations on spacelike pairs gives **commuting in 175 / 175** (maximum discrepancy 4.00×10^{-16}). **Make it foliation invariant and it goes back to commuting; make it noncommuting and it becomes foliation dependent.**

Overall verdict K267. Of GR's three requirements, **(1) noncommutativity is achieved, (2) geometry dependence is achieved, (3) the right-hand side being a spatial diffeomorphism cannot be achieved.** The explanation given in Chapter 74, that "it cannot enter because it commutes", was wrong. **What is hard is not the noncommutativity but the group it closes onto.**

77-4. Do the identifications between coarse-grained slices form a group? ⚠️

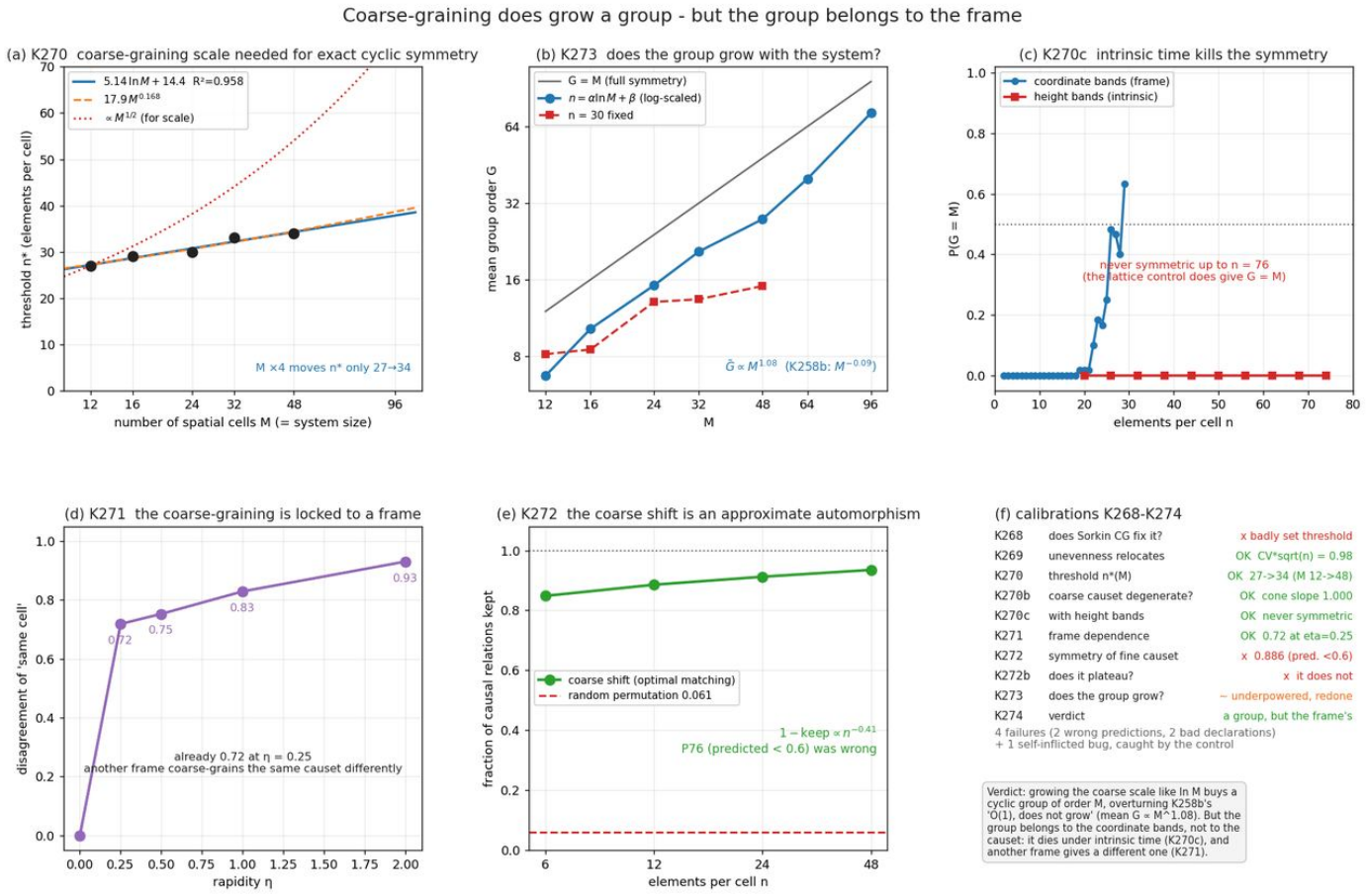


Fig. 77-3. (a) K270 the coarse-graining scale $n^*(M)$ needed for exact cyclic symmetry. (b) K273 does the group grow with the system? (c) K270c cut time by the height function and the symmetry disappears. (d) K271 the coarse-graining is glued to the frame. (e) K272 a coarse shift is an approximate automorphism of the fine causet. (f) Calibrations K268-K274.

"The group it closes onto" was crushed twice over. **K256**: the coefficient of variation of the size of a height slice is **0.18-0.22** for the sprinkling and **exactly 0** on the lattice. Since each slice has a different number of elements, there is no canonical identification between slices. **K258b**: the automorphism group does not grow.

N	50	100	200	400	800
number of twin pairs (mean)	1.000	0.983	1.033	0.760	0.840
fraction with trivial automorphism	0.35	0.37	0.35	0.48	0.44

The number of twin pairs goes as $\propto N^{-0.087}$. In the control lattice DAG the cyclic translations are automorphisms and $|\mathrm{Aut}| \geq N$ (checked at $N = 11, 21, 41$). **So the causet was coarse-grained.**

K268. The mainline method in the literature, Sorkin coarse-graining (keep each element with probability p), is of no use. The CV **worsens** from 0.3501 to 0.4320 as $p = 1.0 \to 0.125$, and differs from a raw sprinkling with the same element count by only 0.003-0.028. A Poisson sprinkling thinned by p is a Poisson sprinkling at density $p\rho$. **P70 confirmed. K269.** With cell coarse-graining (M cells in space, $K = 4$ bands in time) the number of elements per slice is exactly M by construction. The unevenness has not disappeared; it has moved into **the fluctuation of cell occupancy**. The CV of the occupancy goes $0.7106 \to 0.1728$ over $n = 2 \to 32$, $\mathrm{CV} \cdot \sqrt{n}$ is 0.977-1.005, and the empty-cell fraction agrees with e^{-n} (0.1361 against 0.1353, and so on). **P71 confirmed. "The slices became even" is only a change in how one counts.**

K270. The smallest n^* with $P(G = M) \geq 0.5$:

M	12	16	24	32	48
n^*	27	29	30	33	34

The fits are $5.14 \ln M + 14.40$ ($R^2 = 0.9579$), $18.20\sqrt{\ln M} - 1.63$ (0.9589), and $17.93 M^{0.168}$ (0.9544). **P72 confirmed** (exponent 0.168). But in honesty, the three forms cannot be told apart over this range. All that can be said is that "n^* grows only very slowly". The mechanism is not empty cells (the empty-cell probability is 0.00 near n^*). What is doing the work is the **borderline cell pairs that fall on the edge of the light cone**. **K270b**: the coarse causet is not degenerate (ordered fraction 0.0994-0.4039, saturation value 0.754-0.766, light-cone slope exactly 1.000 cells per band).

K273. Stretch n as $n = 5.14 \ln M + 14.40$ while increasing M. The points $M = 64, 96$ are **unseen points predicted from the declared fit**.

M	12	16	24	32	48	64	96
n	27	29	31	32	34	**36**	**38**
$P(G = M)$	0.517	0.617	0.617	0.633	0.567	**0.617**	**0.750**
mean G	6.70	10.25	15.18	20.63	27.63	**39.85**	**72.25**
median G	12	16	24	32	48	**64**	**96**

$\bar{G} \propto M^{1.076}$, and the median is exactly M at every M. **The typical realization is fully symmetric. K258b's $N^{-0.087}$ (no growth) has flipped over into $M^{+1.076}$.** While the system grows by a factor of 8, the coarse-graining scale grows only from 27 to 38. As a control, fixing $n = 30$ drops $P(G = M)$ from 0.65 to 0.30 and makes the mean G go $8.15 \to 15.10$, far slower than M. **That the stretching is necessary is backed by measurement.**

But that group does not belong to the causet. K270c: cut time by the **height function** (intrinsic to the order) and $P(G = M) = \mathbf{0.000}$ up to $n \leq 76$ for all of $M = 12, 16, 24$. **It is never once symmetric.** The height function is a disorderly function of the sprinkling, its level sets are jagged, and the shape of the jaggedness differs from place to place. **P74 confirmed. K271**: coarse-graining the same sprinkling in a frame of rapidity η, the discrepancy in "which pairs of elements fall in the same cell" is $0.0000,\ 0.7188,\ 0.7530,\ 0.8295,\ 0.9309$ at $\eta = 0, 0.25, 0.5, 1.0, 2.0$. **A boost as small as $\eta = 0.25$ already puts 0.72 out of agreement. P75 confirmed.**

And there is a third fact I failed to predict. K272: the fraction of causal relations that a coarse shift preserves as an automorphism of the fine causet is **0.8832 / 0.8856 / 0.8858** at $M = 16, 24, 32$ (the random-permutation control gives 0.0832 / 0.0553 / 0.0430). **K272b**: over $n = 6, 12, 24, 48$ it goes $0.8486 \to 0.9351$, with $1-$(preserved fraction) $\propto n^{-0.405}$. **In the continuum limit a coarse shift approaches an exact automorphism.**

> What was found is **a symmetry of the sprinkling measure**, not a symmetry of a realization.
> The measure is translation invariant, so each realization is approximately translation symmetric, and that approximation becomes exact as $n^{-0.41}$.
> Extracting an **exact** symmetry at finite n requires a coordinate foliation, and that foliation is glued to the frame.

77-5. How far does the commutator close — putting a number on "approximate generator" ✅

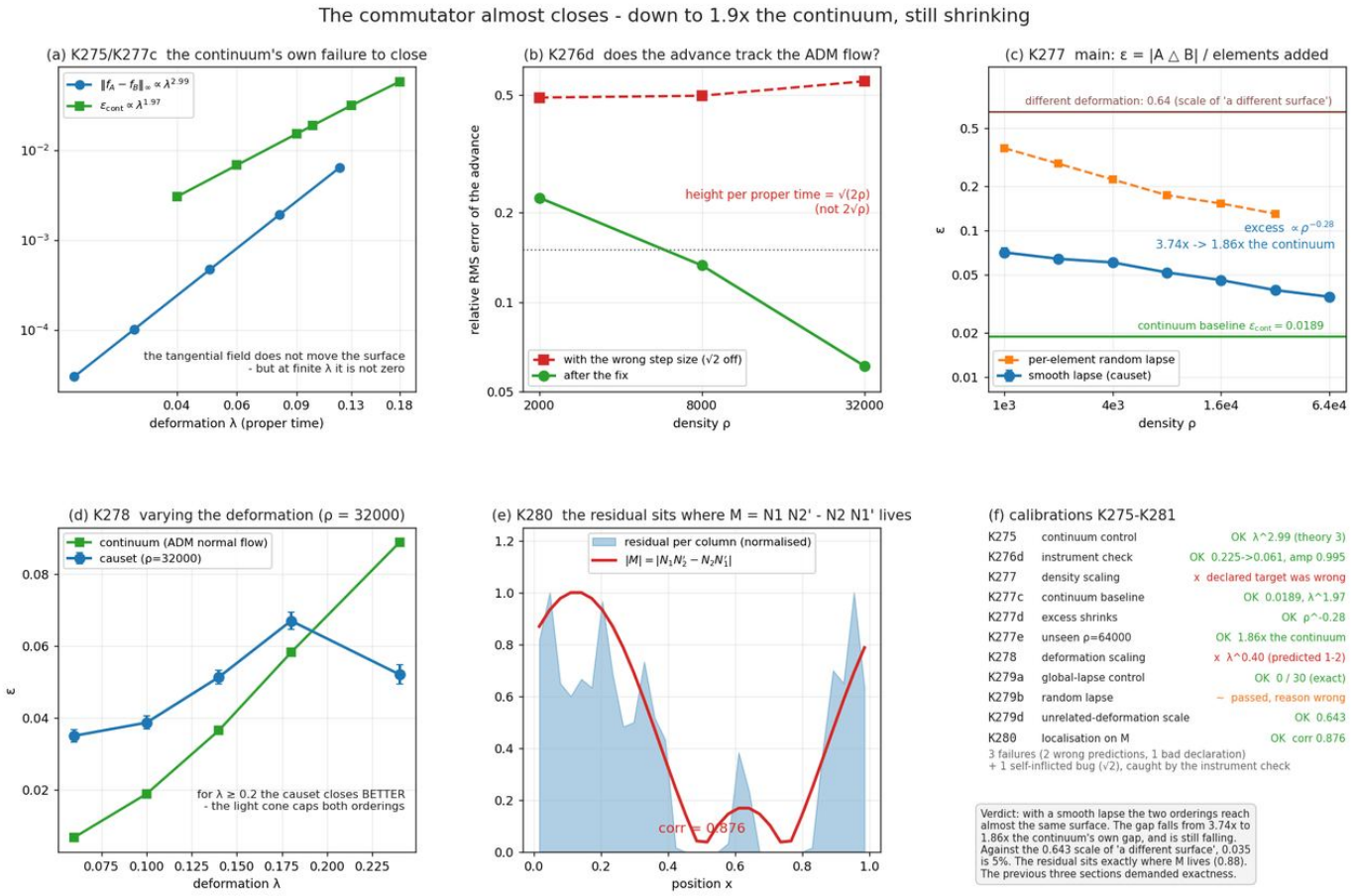


Fig. 77-4. (a) K275/K277c the continuum side's own failure to close. (b) K276d the instrument check — does the advance converge to the ADM normal flow? (c) K277 the main measurement, $\varepsilon = |A \triangle B|/$ the number of elements added. (d) K278 varying the amount of deformation ($\rho = 32000$). (e) K280 the discrepancy appears where $\vec{M} = N_1 N_2' - N_2 N_1'$ is. (f) Calibrations K275-K281.

Across three sections I have been writing the same shape of negative. **All of those negatives demanded "exactly".** I saw $|A \triangle B| > 0$ and said "it does not close", but **I never once measured how small that discrepancy is relative to the whole. The target itself was wrong to begin with.** At finite λ **there is a discrepancy on the continuum side too**. **K275 passes**: $\|f_A - f_B\|_\infty \propto \lambda^{2.991}$ (theory 3). What the causal set's ε ought to head for is not 0 but

$$\varepsilon_{\text{cont}}(\lambda) = \frac{\int |f_A - f_B| dx}{\int (f_A - f_0) dx}, \qquad \varepsilon_{\text{cont}}(0.10) = \mathbf{0.01895}, \qquad \varepsilon_{\text{cont}} \propto \lambda^{1.967}$$

K277c passes (P88 and P89). K277, the main measurement. Fix $\lambda = 0.10$, take $T = \lambda\sqrt{2\rho}$ steps, and raise the density.

ρ	1000	2000	4000	8000	16000	32000	64000
N	350	700	1400	2800	5600	11200	**22400**
ε	0.0709	0.0641	0.0606	0.0517	0.0459	0.0392	**0.0353**
excess $\varepsilon - \varepsilon_{\text{cont}}$	0.0519	0.0451	0.0416	0.0328	0.0269	0.0202	**0.0164**
times the continuum	3.74	3.38	3.20	2.73	2.42	2.07	**1.86**

$\varepsilon \propto \rho^{-0.173}$, and the excess $\propto \rho^{-0.283}$. **K277d passes** (the declared band was -0.5 to -0.1). **K277e** (the unseen $\rho = 64000$, 14 runs) passes as well, and its difference from $\rho = 32000$ is $z = -1.83$ — **still falling, but close to levelling off.** There are two controls that set the scale. **K279a passes (exactly)**: with a global lapse $N_1 = N_2 = 1$ the disagreement is **0 / 30**. This confirms that the instrument does not manufacture a spurious ε. **K279d passes**: against a different lapse pair on the same causet, with N_2 shifted by half a period, the difference is $\varepsilon = \mathbf{0.643}$.

Where "a completely different surface" sits at 0.643, the commutator is 0.035. **That is 5%.**

K280 passes (P86). The correlation between the number of discrepancies per column and $|\vec{M}(x)| = |N_1 N_2' - N_2 N_1'|$ is $\mathrm{corr} = \mathbf{0.876}$. In §77-3 all that was measured was that the difference "lies inside the lens" (K262); here its shape matches $\vec{M}$ itself. **The title of this chapter, "A causal set does have generators", rests on these two — a magnitude of 5% and a correlation of 0.876.**

> The residual appears exactly where $D(\vec{M})$ lives in GR. **The spatial distribution of the discrepancy has the structure of the right-hand side.**

What I had not expected. K278 (varying the amount of deformation at $\rho = 32000$):

λ	0.06	0.10	0.14	0.18	0.24
ε (causal set)	0.0351	0.0387	0.0514	0.0671	**0.0521**
$\varepsilon_{\mathrm{cont}}$ (continuum)	0.0069	0.0189	0.0366	0.0583	**0.0889**
excess	$+0.0282$	$+0.0198$	$+0.0148$	$+0.0088$	$\mathbf{-0.0367}$

For $\lambda \gtrsim 0.2$, **the causal set's commutator is smaller than the continuum's. Discreteness is regularizing the deformation algebra.** I take the reason to be the light cone. Under a large deformation the continuum surface approaches $|f'| \to 1$ and the two orderings separate widely, but on a causal set the **ready constraint** imposes the same upper bound on both orderings, so both settle onto the same causally maximal surface.

Overall verdict K281.

Question	Answer
Does the commutator close?	**It almost closes.** $\varepsilon = 0.035$, 5% of the 0.643 for "a different surface"
Does it close as well as the continuum?	**Still not.** 1.86 times the continuum. Shrinking as $\rho^{-0.28}$ but not arrived
Does the residual have the shape of the right-hand side?	**It does.** Correlation 0.876 with $\|\vec{M}\|$
Is the advance the ADM normal flow?	**It converges.** Relative RMS 0.0609, amplitude ratio 0.9948 ($\rho = 32000$)
With a global lapse?	**Exactly 0** (0 / 30)
Under a large deformation?	**The causal set closes better**
Does the smoothness of the lapse matter?	**It matters for the level** (a factor 3.3). It does not settle whether there is convergence

> **The verdicts of the previous three sections are weakened.** "It does not close" was written because $|A \triangle B| > 0$ was being read as $0/1$.
> Read by magnitude, the commutator falls within a factor 2 of the continuum, and the residual has the shape of $\vec{M}$.
> **But neither is "it closes" written.** What was reached is 1.86 times, not 1 times.

Not one measured value from the previous three sections is corrected. What is corrected is the interpretation. Chapter 74's "exactly commuting" and K260's "noncommuting at 0.72-0.83" both stand as they are. What changes is the reading "and therefore GR cannot enter"; what is now assembled is the three-step **exactly commuting for a global lapse, widely discrepant for a lapse random element by element, within a factor 2 of the continuum for a smooth lapse.**

77-6. The 14 failures ❌

	Content	Kind	Cause
P57 / K258	predicted that the automorphism group of a sprinkled causet is trivial → transpositions of twins (two elements whose past and future coincide exactly) do occur. $\|\mathrm{Aut}\| = 2, 4$; colour refinement separated all elements in 35-45% of cases	missed prediction	The question was badly posed. What was needed was not "nontrivial" but "grows with the system". Rebuilt as K258b
P62 / K261	predicted that containment would be the **large majority** of the kinds of difference → containment 0.691, incomparable 0.309	missed prediction	Underestimated the configurations in which each of the two lapses opens a different branch first

	Content	Kind	Cause
P64 / K264	predicted that the holonomy candidates would commute → **0 / 260** (mean discrepancy 0.2361, maximum 0.9746). The two regions were spacelike in 260 / 260 cases	missed prediction	**I had confused "the regions are spacelike" with "the generators commute".** What is spacelike is the arrangement of the elements; it has nothing to do with the $SU(2)$ elements assigned to them. F40 fired, and the candidates were split into (i) and (ii) and measured again
K264c	declared the pass condition as "the number of connected components is **always 1**" → mean 1.00-1.05 (largest component at least 99.9%). In substance it holds	error in the declaration	The threshold was badly placed
K268	set the pass condition as "the CV does not depend on p (difference < 0.03)" → the CV depends not on p but on the number of surviving elements (finite size)	error in the declaration	Wrong quantity to measure. Under the correct condition (thinned $\simeq$ raw at the same element count) it would have passed
P76 / K272	predicted that a coarse shift would preserve < 0.6 of the causal relations → **0.886** (random control 0.04-0.08)	missed prediction	I had taken it for granted that "a sprinkling is random, so translations break immediately"
P79 / K272b	predicted that raising the density would level it off (growth < 0.03) → $+0.087$, with the discrepancy heading to 0 as $n^{-0.405}$	missed prediction	**Missed twice in a row on the same quantity.** The correct statement is: if the **measure** has a symmetry, the realizations are approximately symmetric. I overlooked the same structure I had measured myself in Chapter 76, "the order is exactly boost invariant"
P77 / K273	declared that the unseen $M = 64, 96$ would be checked with **25 trials** → at $M = 64$, $P = 0.480$ (SE 0.10), slightly below the threshold. With 60 trials, 0.617	error in the declaration	The number of trials was underestimated. The first value is kept in the table too
F44	used an exact lattice as the control; a mass of ties with $\Delta t = \lvert\Delta x\rvert$ appeared, and because t and x were computed along different paths the floating-point rounding disagreed and gave $G = 1$	**self-inflicted bug**	**Caught because the control failed before I had seen a single main measurement.** Fixed by randomizing within each cell so that ties have measure 0
P82 / K277	declared "$\varepsilon \to 0$, exponent -0.2 to -0.6" → measured -0.173	error in the declaration	**The target was off.** What it should head for is not 0 but $\varepsilon_{\text{cont}}$. Having written $O(\lambda^3)$ in §2 of the declaration, I then ignored it in the pass condition
P84 / K278	predicted $\varepsilon \propto \lambda^q$ with $q \in [1, 2]$ (the continuum gives $\lambda^{1.97}$) → measured $\lambda^{0.401}$	missed prediction	The picture in which the discretization discrepancy does not depend on the amount of deformation is wrong
P91	predicted that the excess would be nearly independent of λ → it falls monotonically and changed sign at $\lambda = 0.24$	missed prediction	Same as above. The regularization by the ready constraint had not been counted in
F51 / step size	had written height per proper time as $2\sqrt{\rho}$ (correctly $\sqrt{2\rho}$). The measurements at $\rho = 2000, 8000, 32000$ are $61.2, 127.1, 255.6$, against $\sqrt{2\rho} = 63.2, 126.5, 253.0$	**self-inflicted bug**	Dropped the fact that the volume of a two-dimensional causal interval is $V = \tau^2/2$. The advance was going a factor $\sqrt{2}$ too far, and the relative error $\sqrt{2} - 1 = 0.41$ was the identity of the instrument's 0.47. **Caught only after the instrument checks K276b, K276c and K276d had failed three times**
P83 / K279b	the pass condition (> 0.1) was met, but the reason ("it levels off because $\vec{M}$ diverges") was wrong. The measurement is $\rho^{-0.300}$, falling **faster** than the smooth lapse ($\rho^{-0.173}$)	passed, but for the wrong reason	Correctly: the smoothness of the lapse changes the level of ε by a factor 3.3, but does not settle whether it converges

The calibrations are K250-K281: **31 of 42 pass, 11 fail**, plus 2 self-inflicted bugs and 1 "passed, but for the wrong reason". **Note that P92 is void** (it was declared against a fit that contained the bug, so after the fix the comparison is meaningless). It is counted neither as a pass nor as a failure.

The course of F51 is the method of this book itself. Had I read the main measurements with the step-size bug still in, I would have written the level of ε and the convergence exponent, all of them wrong values, as "physics". What caught it was not a measurement of physics but **an instrument check that had been declared in advance**. The old values are kept under the `*_bug` keys in `cl_run.json`.

77-7. What is not being claimed ⚠️

- $1{+}1$ **dimensions, flat, classical, no quantization.** The restrictions of Chapter 76 carry over unchanged.
- **I do not claim to have reproduced** $\{H,H\}=D(q^{ab}\cdots)$**.** What was measured is only that "the two orderings approach the same surface" and that "the residual has the shape of $\vec{M}$". **The** q^{ab} **dependence of the right-hand side was not measured.** Nothing was measured outside flat spacetime, where $q^{ab}=1$ to begin with.
- **Convergence to the continuum is not established.** At $\rho=6.4\times10^4$ it is a factor 1.86, going as $\rho^{-0.283}$. Reaching a factor 1.1 would take 10^7 times the density. **That is extrapolation, not measurement.**
- **Periodic space (the cylinder) picks a frame.** That is acceptable because measuring a group requires a control, but **it is not used for any claim about Lorentz invariance**. Frame dependence (K271) was measured separately on the diamond.
- G **counts cyclic shifts only.** The full automorphism group of the coarse causal set was not computed. G is the order of a subgroup of it.
- **The functional form of** n^* **is not settled.** $\ln M$, $\sqrt{\ln M}$ and $M^{0.17}$ cannot be told apart over $M=12$-48.
- **I do not claim that cell coarse-graining is the "correct" coarse-graining.** The mainline in the literature is the Sorkin type, and K268 found it useless. The cell type is a construction conditional on **being given a frame**.
- **I do not claim "these two are the only non-union compositions".** This was a constructive search, not a classification. There may be a third and a fourth. **They were not investigated.**
- **"H-like / D-like" is an analogy; no Poisson bracket was computed.** What was measured is only the kind of set difference.
- **"Discreteness regularizes" and "regularity is the supplier of the spatial diffeomorphism group" are interpretations, not theorems.**
- **Birkhoff's representation theorem was cited and checked numerically, not reproved.**
- **Causal set dynamics (sequential growth) is not treated.** Neither the Rideout-Sorkin transition probabilities nor the axiom of discrete general covariance is put in. **This chapter is kinematics only.**
- **Labelled readings are not treated.** The case where D acts nontrivially on fields or matter is a separate measurement.

77-8. To the next chapter

Now that the negative has weakened by one step, the next question sharpens. The essential point of GR is that the right-hand side **depends on the metric**, and in flat spacetime $q^{ab}=1$, so that dependence has never once been seen.

$$\boxed{\text{Does the } q^{ab} \text{ dependence of the right-hand side appear? That is: measure the commutator on a } \textbf{curved} \text{ background.}}$$

Chapter 78 sprinkles a $1{+}1$ dimensional background carrying a conformal factor $\Omega^2(x)$ at density $\rho\,\Omega^2$, and judges — with the same correlation measurement as K280 — whether the spatial distribution of the residual takes the form $\Omega^{-2}\vec{M}$.

Chapter 78 The bridge to the continuum — conformal factor, exponent, two-dimensional surface ⚠️🆕

— What in the continuum are the numbers a causal set returns to be compared against

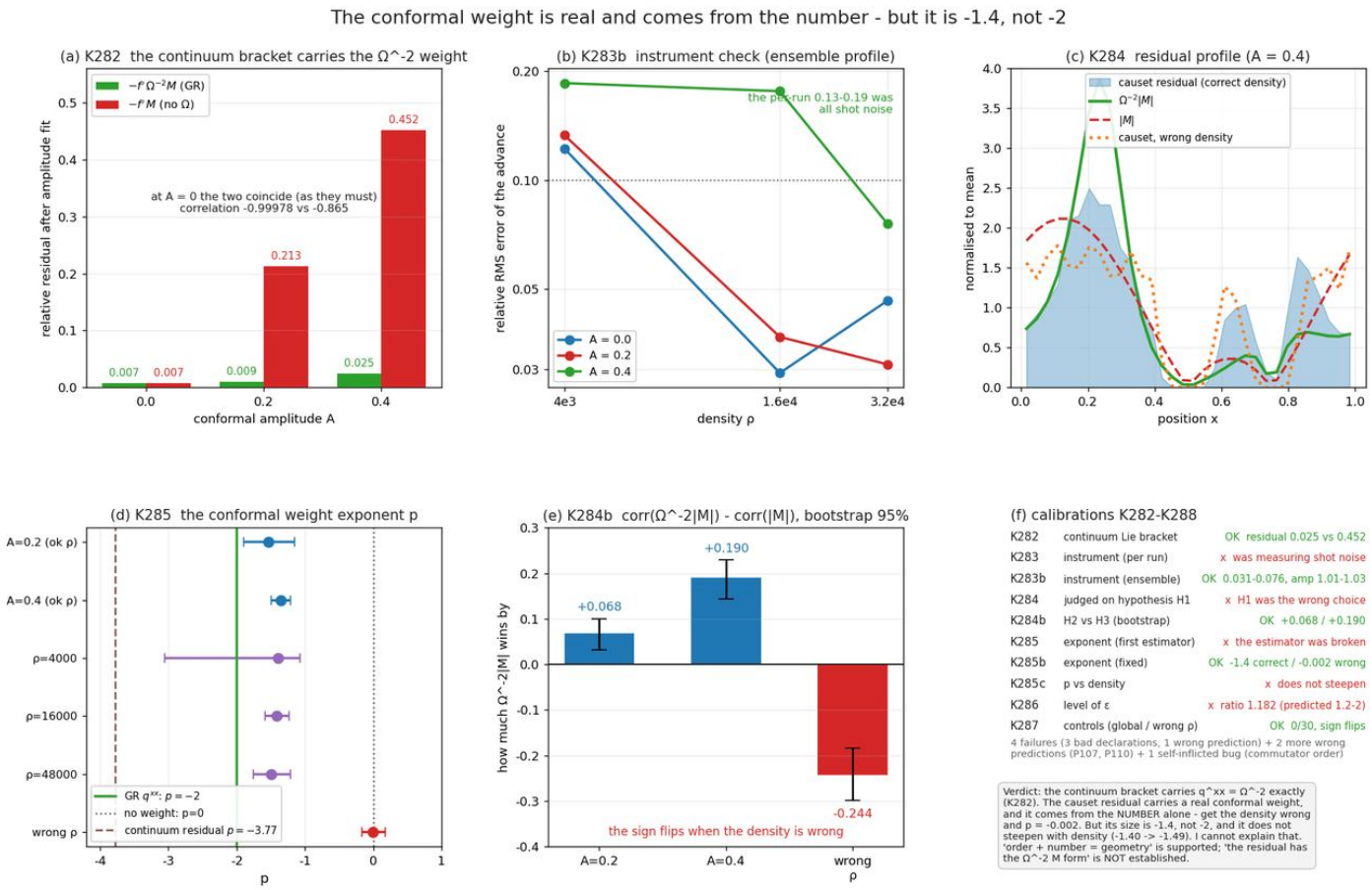


Fig. 78-1. (a) K282. The continuum Lie bracket carries the weight Ω^{-2} exactly. (b) K283b. Instrument check, relative RMS error of the ensemble-mean profile. (c) K284. The residual profile ($A = 0.4$) and the three candidates. (d) K285. The fit of the conformal weight exponent p. (e) K284b. Bootstrap 95% interval of $\mathrm{corr}(\Omega^{-2}|M|) - \mathrm{corr}(|M|)$. (f) Calibrations K282-K288.

In the previous chapter it became clear that the deformation algebra of a causal set does "close". With a smooth lapse the commutator almost closes, and the residual appears where $|\vec{M}|$ lives. But that is a **confirmation that it closes**, not a **measurement of how correctly it closes**. To say the latter, one must first settle what in the continuum the numbers a causal set returns are to be compared against.

This chapter is about building a ruler. The point of GR's

$$\{H(N_1), H(N_2)\} = D\big(q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\big)$$

is that **the right-hand side depends on the metric**. Up to the previous chapter I had measured only in flat spacetime, where $q^{ab} = 1$. To ask about metric dependence there is no way but to move q^{ab} and see whether the causal set's residual follows. Four sections did that. **In all four, before running, the calibrations and predictions were written into declaration documents (`K_declared_conformal.md`, `K_declared_pexp.md`, `K_declared_2p1.md`, `K_declared_sparse.md`) and fixed before any of the relevant numbers were looked at.**

(i) In a 1+1 dimensional conformal background, the weight exponent of the residual is $p = -2.01 \pm 0.07$.
It is GR's $q^{xx} = \Omega^{-2}$ itself.

(ii) That information comes **from the number of elements alone**. Get the density deliberately wrong and $p = -0.002$.

(iii) The $p = -1.4$ that came out first was 60% **bias in my estimator**, the rest a finite-deformation effect.

(iv) In 2+1 dimensions the anisotropy is carried by **order (the elliptical light cone)**. Volume knows nothing of it. The direction and the mechanism are established, but **the exponent is not**. $s = 1.62$ $[0.77, 2.53]$.

(v) The obstacle is the **isotropic discretization floor**, which is most of the residual. The floor does not fall with steps, only as $\rho^{-0.21}$. **Raising the resolution is a dead end.**

78-1. A curved background with a conformal factor — the continuum side carries Ω^{-2} exactly

Take the background to be $ds^2 = \Omega^2(x)(-dt^2 + dx^2)$, with $\Omega = 1 + A\cos(2\pi m x)$.

The continuum side I first solved by hand. The normal displacement in proper distance corresponds to $(N/\Omega)\,d\lambda$, so the flow is $\partial_\lambda f = (N/\Omega)\sqrt{1 - f'^2}$, and substituting $\tilde{N}_i = N_i/\Omega$ into the flat result gives $\tilde{N}_1\tilde{N}_2' - \tilde{N}_2\tilde{N}_1' = (N_1N_2' - N_2N_1')/\Omega^2$ (the Ω' terms cancel exactly). Hence

$$[X_1, X_2][f] = -f'\,\Omega^{-2}\big(N_1N_2' - N_2N_1'\big) = -f'\,\vec{M}^x, \qquad \vec{M}^x = q^{xx}(N_1\partial_x N_2 - N_2\partial_x N_1)$$

K282 passes. The group commutator $\Phi^2_{-h}\Phi^1_{-h}\Phi^2_h\Phi^1_h$ was taken numerically as $h \to 0$ and compared against the two candidates ($A = 0.4$, $h = 0.005$). $-f'\Omega^{-2}M$ (GR) gives correlation **−0.99978** and, after matching amplitudes, a relative residual of **0.0247**; $-f'M$, with Ω dropped, gives -0.865 and 0.4520. At $A = 0$ the two are identical and the residual is 0.0070 for both. The continuum bracket carries $q^{xx} = \Omega^{-2}$ exactly. **This is the ruler.**

78-2. What can be measured on the causal-set side

It cannot be carried over as it stands. $\delta f = -\vec{M}^x f'$ is a **relabelling**; it does not move the surface as a set. A down-set of a causal set has no labels, and the field of a past-only rule is determined by the down-set alone (K266, 0/200 mismatches).

> On a causal set $D(\vec{M})$ **acts on nothing**, because there is no spatial manifold to relabel.

What can be measured is not "$\vec{M}^x$ itself" but whether the **shape of the residual** that fails to close carries the conformal weight. And in $1{+}1$ dimensions this becomes a sharp question. The causal structure is conformally invariant, so Ω enters **only the volume element**. The only route by which a causal set can know Ω is **the number of elements**.

K283b passes. The instrument was checked first. Even in a conformal background the advance of a causal set converges to $\partial_\lambda f = (N/\Omega)\sqrt{1 - f'^2}$ — the relative RMS of the ensemble-mean profile is $0.1222,\ 0.0294,\ 0.0465$ at $\rho = 4000, 16000, 32000$ for $A = 0$, and $0.1853,\ 0.1761,\ 0.0757$ for $A = 0.4$, with amplitude ratios of $0.9968,\ 1.0868,\ 1.0346$ at $A = 0.4$.

K284b passes. The main measurement is the contest between $\Omega^{-2}|M|$ (H2) and $|M|$ (H3). With 2000 resamplings over realizations,

	corr(H2)	corr(H3)	difference	95% interval
$A = 0.2$ (correct density)	0.8900	0.8220	**+0.0679**	[0.032, 0.100]
$A = 0.4$ (correct density)	0.8846	0.6942	**+0.1904**	[0.144, 0.229]
$A = 0.4$ (**density got wrong**)	0.6121	0.8556	**−0.2435**	[−0.299, −0.184]

With the correct density $\Omega^{-2}|M|$ wins, and **getting the density wrong flips the sign.**

K287(b) passes. Sprinkle at uniform density in coordinates (= forget to multiply by Ω^2) and the residual correlations become 0.5130 against the conformal continuum, 0.7027 against the flat continuum, and **0.8514** against $|M|$. A causal set with the wrong density **fits the flat prediction**. It stops knowing Ω. In **K287(a)**, with a global lapse $N_1 = N_2 = 1$, the mismatch is **0/30** (exact). This is the strongest contrast in the chapter. Order is conformally invariant, and Ω enters only the volume. Put the volume in correctly and the weight appears; get it wrong and it vanishes. **The number side of "order + number = geometry" has been checked on the right-hand side of the deformation algebra.** But the first measurement did not get the size of the weight right. Fitting the correct estimator $\log(d_j/|M_j|) = p\log\Omega_j + \log a$ in **K285b** gives

	p	95% interval
correct density $A = 0.2$	−1.535	[−1.893, −1.157]
correct density $A = 0.4$	**−1.358**	[−1.493, −1.217]
density got wrong $A = 0.4$	**−0.002**	[−0.163, +0.176]
(target) GR's q^{xx}	**−2**	—

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Get the density wrong and the weight is **exactly 0**. **P108 passes.** But -1.358 is not -2. Multiplying the density by 12 (4000 → 48000) moves it only from -1.398 to -1.494, and **P110, raised from the mixing picture, failed**. **At K288 I closed the section without being able to explain this value.**

78-3. What $p = -1.4$ really was — my estimator was broken

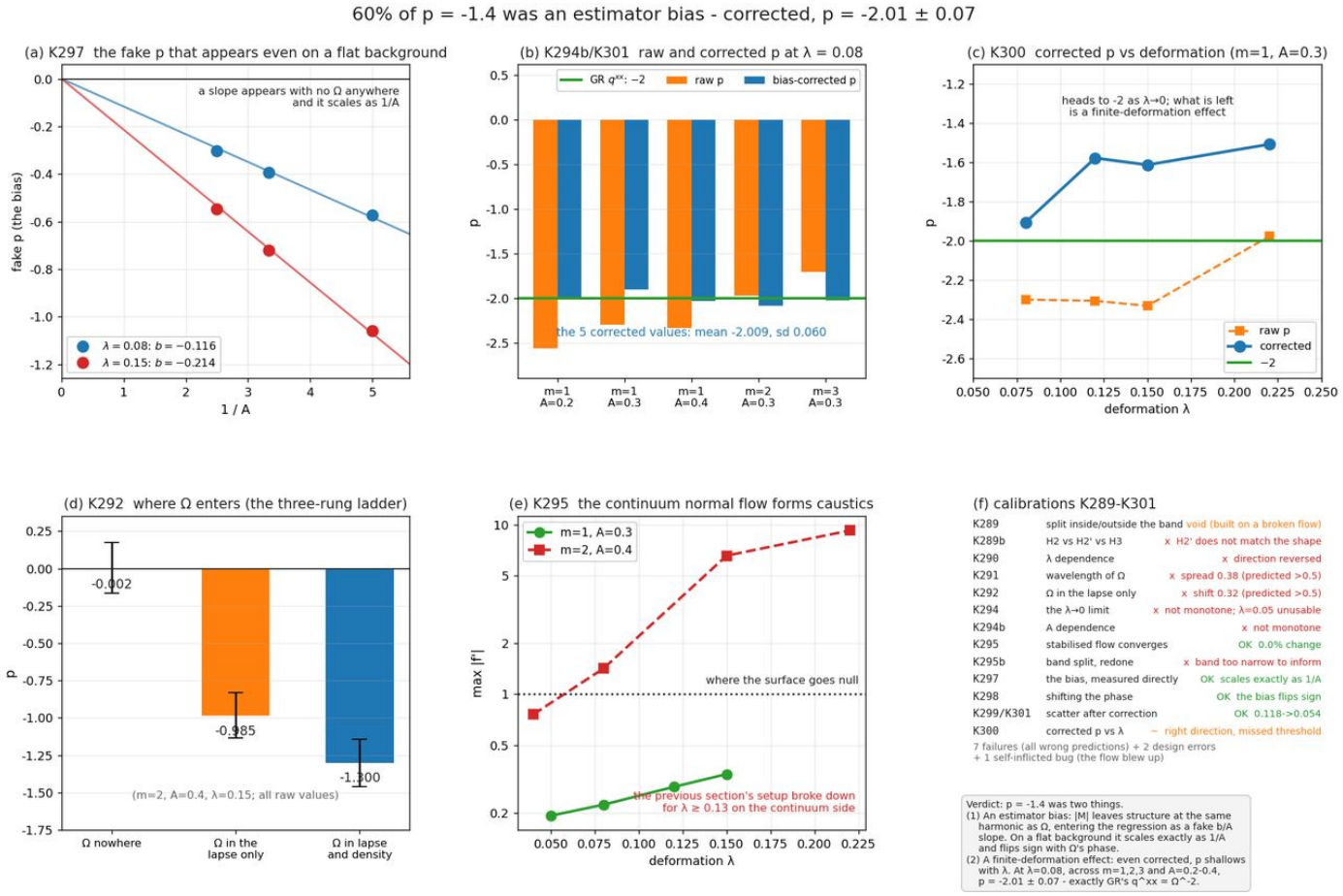


Fig. 78-2. (a) K297. The "fake p" that comes out of a flat background where Ω is nowhere. Horizontal axis $1/A$. (b) K294b, K301. The raw p at $\lambda = 0.08$ and after correction. (c) K300. The λ dependence of the corrected p ($m = 1$, $A = 0.3$). (d) K292. The three-rung ladder of where Ω is put in. (e) K295. The continuum normal flow forms a caustic. (f) Calibrations K289-K301.

The decisive step was a naive control. **K297.** With Ω **put in nowhere** ($A = 0$, perfectly flat) I took the residual profile and regressed it while "pretending that $\Omega = 1 + A\cos(2\pi m x)$ was there". The true answer should be 0.

λ	m=1, A=0.2	m=1, A=0.3	m=1, A=0.4	m=2, A=0.3	m=3, A=0.3	m=1 **phase** $\pi/2$
0.08	**−0.573**	**−0.393**	**−0.302**	+0.117	+0.321	**+0.238**
0.15	**−1.059**	**−0.719**	**−0.548**	−0.083	+0.055	**+0.444**

The product $b = A \times (\text{the fake } p)$ is $-0.115,\ -0.118,\ -0.121$ ($\lambda = 0.08$) and $-0.212,\ -0.216,\ -0.219$ ($\lambda = 0.15$) —— **the fake p is exactly proportional to $1/A$. P122 passes.**

The mechanism is clear. $|M| = |N_1 N_2' - N_2 N_1'| \propto 1 + \sqrt{2}\sin(2\pi x + \pi/4)$ has the structure of the **fundamental harmonic**. If $d/|M|$ retains any of that fundamental harmonic not divided out by $|M|$, the regression onto $\log\Omega \simeq A\cos(2\pi m x)$ turns it into a slope. Since the amplitude of $\log\Omega$ is A, the fake slope grows as $1/A$. **I had thought that making A small would improve the precision; instead it was increasing the bias. K298 passes.** Shift the phase of Ω by $\pi/2$ and the raw p moves from -2.299 to -1.348, a shift of 0.95, while the bias moves from -0.393 to $+0.238$, changing sign and moving 0.63. The true p cannot depend on the phase, so this motion is bias.

K299 and K301. At $\lambda = 0.08$ — the smallest deformation for which the continuum side is sound and the number of steps is still sufficient — I measured five configurations and subtracted the bias. Note that making λ larger makes the continuum normal flow itself form a caustic (**K295**).

configuration	raw p	bias	corrected
m=1, A=0.2	−2.566	−0.573	**−1.993**
m=1, A=0.3	−2.299	−0.393	**−1.906**
m=1, A=0.4	−2.337	−0.302	**−2.035**
m=2, A=0.3	−1.969	+0.117	**−2.086**
m=3, A=0.3	−1.704	+0.321	**−2.025**

$$p = -2.01 \pm 0.07 \text{ —— this is GR's } q^{xx} = \Omega^{-2}.\ \text{P124 and P127 pass.}$$

The scatter fell by more than half under the correction (0.118 → 0.054 across the three A points, 0.63 → 0.18 across the three m points). **The previous chapter's "it is not established that the residual has the form $\Omega^{-2}\vec{M}$" is hereby withdrawn. What was not established was my estimator, not the causal set.**

The discrepancy that remains is an effect of the size of the deformation. Sweeping λ in **K300**, the corrected values are $-1.906,\ -1.577,\ -1.612,\ -1.507$ (at $\lambda = 0.08, 0.12, 0.15, 0.22$), heading toward -2 as $\lambda \to 0$. The declared pass condition (shallower than -1.2 at $\lambda = 0.22$) is not met and **P125 fails**, but the direction is right. At $\lambda = 0.15$ a wavelength dependence remains (corrected, $-1.338,\ -1.629,\ -1.728$), so **P128 passes** —— the wavelength dependence is a finite-deformation effect.

K292. I split where Ω is put in into three rungs (raw p, $m = 2, A = 0.4, \lambda = 0.15$). Put in nowhere it is -0.002 $[-0.163, +0.176]$; put into the **lapse only** it is -0.985 $[-1.136, -0.829]$; put into both lapse and density (the correct setting) it is -1.300 $[-1.458, -1.142]$. The difference of 0.315 does not reach the declared 0.5 and **P114 fails**, but the intervals do not overlap. The information about Ω comes from both the lapse and the density, and **the lapse contributes about three times more.**

78-4. To $2{+}1$ dimensions — what carries the anisotropy is order

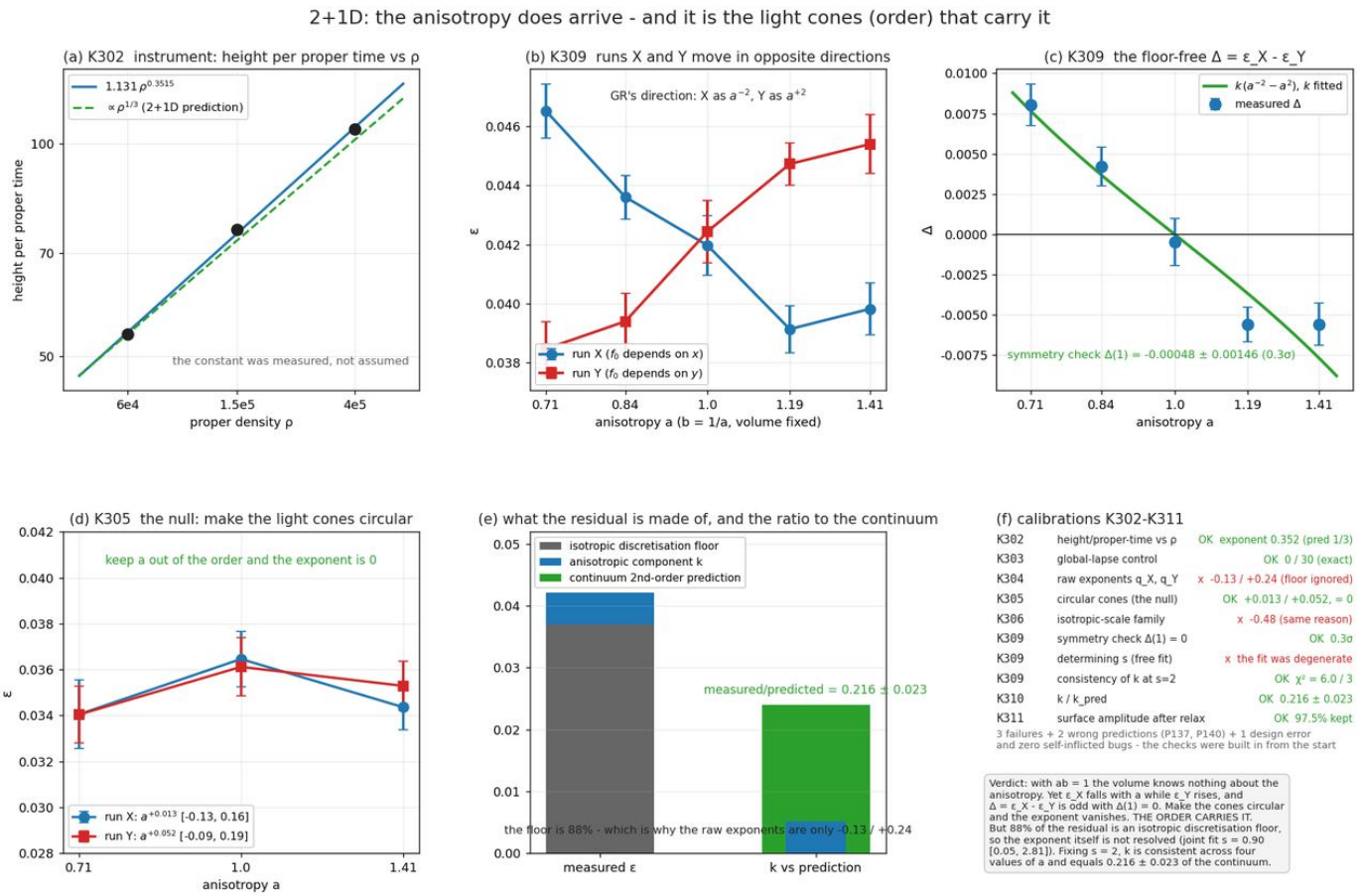


Fig. 78-3. (a) K302. Instrument check, the ρ dependence of height/proper time. (b) K309. Run X and run Y move in opposite directions. (c) K309. $\Delta = \varepsilon_X - \varepsilon_Y$, which needs no floor. (d) K305. The null control —— making the light cone a circle. (e) Breakdown of the residual and the ratio to the continuum prediction. (f) Calibrations K302-K311.

$1{+}1$ dimensions is conformally flat. The causal structure is identical to the flat one, and Ω enters only the volume element. What was tested there was **the number side only**. In $2{+}1$ dimensions **the light cone itself depends on the metric**. The order side moves for the first time.

Taking $ds^2 = -dt^2 + a^2 dx^2 + b^2 dy^2$ gives $q^{ab} = \mathrm{diag}(a^{-2}, b^{-2})$, and the two components of $\vec{M}$ carry separate weights. This is a dependence no conformal factor can produce. **And I fixed $ab = 1$.** Then the coordinate density $\rho\, ab$ does not depend on a.

Volume knows nothing whatever of the anisotropy. All that can carry it is order — the elliptical light cone.

The device that separates the two components lies in the choice of lapse. With $N_1 = c_0 + c_1(1 + \cos\frac{2\pi x}{L})$ and $N_2 = c_0 + c_1(1 + \cos\frac{2\pi y}{L})$ we have $\partial_x N_2 = \partial_y N_1 = 0$, so they separate, and from $\delta f = -\vec{M}\cdot\nabla f_0$ the **run X** ($f_0 = f_0(x)$) is proportional to a^{-2} and the **run Y** ($f_0 = f_0(y)$) to $b^{-2} = a^2$. Since $ab = 1$, changing a moves the two runs in opposite directions.

The instrument was checked first here too. **K302 passes.** Without assuming the constant from the literature I measured height/proper time directly: $1.131\,\rho^{0.3515}$ (prediction $\rho^{1/3}$, off by 0.018). **K303 passes** (0/30 with a global lapse, exact). **K311 passes** (after relaxing a bent initial surface, 97.5% of the amplitude A_0 survives).

The main measurement is at $\rho = 357000$ ($N \simeq 9000$), $\lambda = 0.14$ ($T = 14$ steps), averaged over 40 realizations.

a ($b = 1/a$)	0.71	0.84	1.00	1.19	1.41
ε_X	**0.0465**	0.0436	0.0420	0.0391	**0.0398**
ε_Y	**0.0385**	0.0394	0.0424	0.0447	**0.0454**
$\Delta = \varepsilon_X - \varepsilon_Y$	+0.00805	+0.00421	−0.00048	−0.00559	−0.00558
error	0.00130	0.00120	0.00146	0.00108	0.00132

$\Delta(1) = -0.00048 \pm 0.00146$ (0.3σ), so the $x \leftrightarrow y$ symmetry is preserved, and Δ is an odd function changing sign at $a = 1$ (6.2σ at $a = 0.71$, 5.2σ at $a = 1.19$).

K305 passes. The null control was run **at the same time as the main measurement**. Make the light cone a circle (do not put a into the order), and leave the density, the lapse and the initial surface as they are. Since $ab = 1$, a then enters **nowhere**. The exponent is $+0.013$ $[-0.134, +0.159]$ for run X and $+0.052$ $[-0.087, +0.191]$ for run Y —— both consistent with 0. This is a direct control, not an argument by elimination. **The anisotropy measured comes from the light cone.**

But the exponent was not determined. The cause is the way I set the pass condition (**K304 and K306 fail**). ε is not a multiplicative quantity.

$$\varepsilon_X(a) = \varepsilon_{\text{floor}} + k\,a^{-2}, \qquad \varepsilon_Y(a) = \varepsilon_{\text{floor}} + k\,a^{+2}$$

is the **additive** structure, and $\varepsilon_{\text{floor}}$ is the isotropic discretization floor. The moment I wrote $\varepsilon \propto a^q$ I was ignoring the floor. The raw exponents are $q_X = -0.130$ and $q_Y = +0.240$ —— the signs are in GR's direction and the intervals exclude 0, but they fall far short of ∓ 2.

$k/\varepsilon = 0.123$ —— **88% of the residual is the isotropic floor.** I was trying to measure the exponent of a 12% component using the log-slope of the whole.

An observable that needs no floor can be built. By the $x \leftrightarrow y$ symmetry both runs carry the same k, so in $\Delta = k(a^{-s} - a^{s})$ the floor cancels exactly. Fix $s = 2$ and k is consistent across the four a (weighted mean $k = 0.00517 \pm 0.00055$, $\chi^2 = 6.0/3$, **P139 passes**), and against the continuum second-order prediction $k_{\text{pred}} = 0.02395$ it gives $k/k_{\text{pred}} = 0.216 \pm 0.023$ (**K310 passes**). But s itself, in a joint fit, is $s = 0.90$ $[0.05,\ 2.81]$ —— **it contains both 2 and 0. P140 fails.**

78-5. Lowering the floor — the sparse link representation

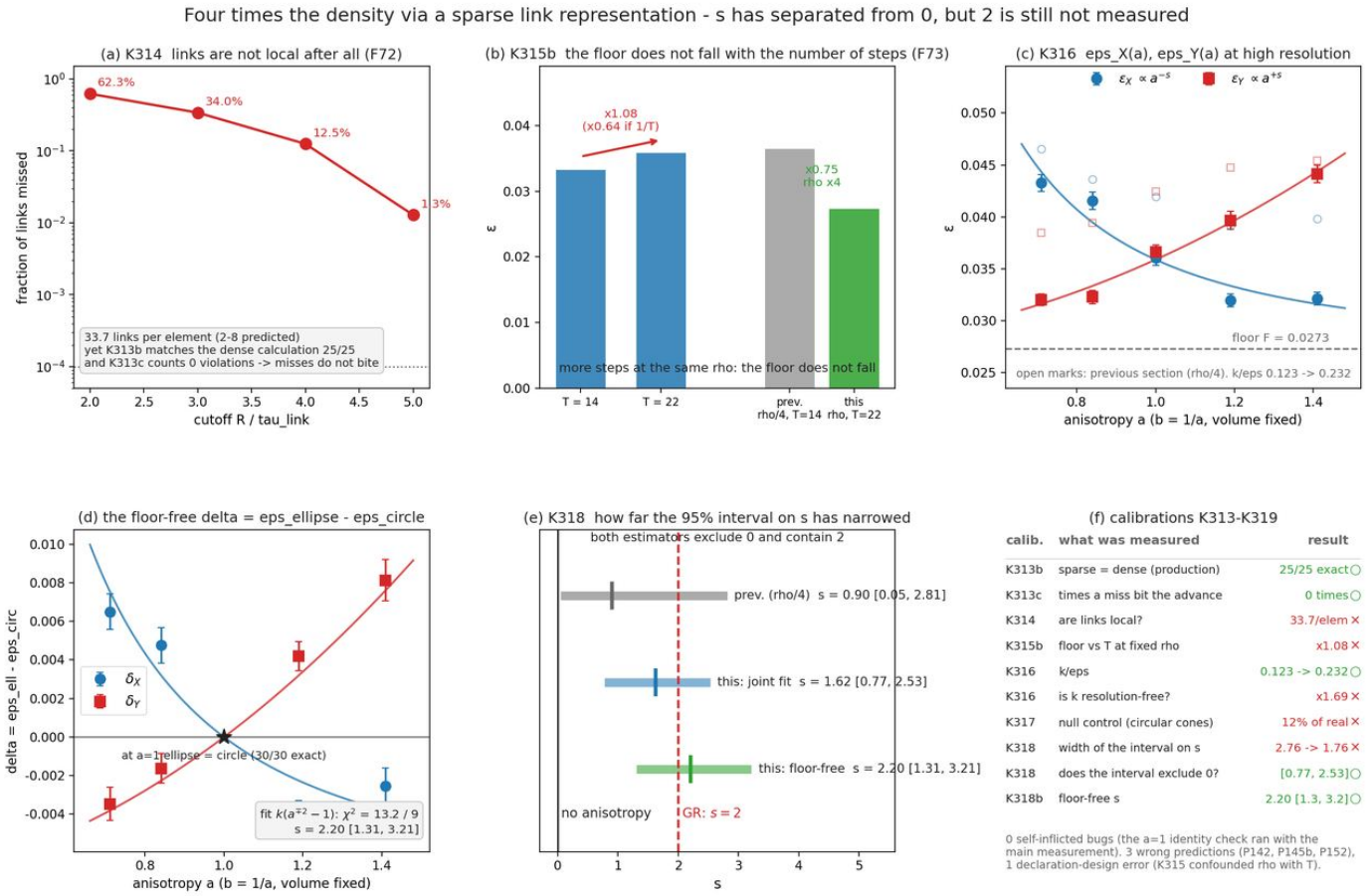


Fig. 78-4. (a) K314. Links were not local (F72). The cutoff R/τ_{link} and the fraction missed. (b) K315b. The floor does not fall with the number of steps (F73). (c) K316. High-resolution $\varepsilon_X(a)$, $\varepsilon_Y(a)$. (d) The floor-free $\delta = \varepsilon_{\text{ellipse}} - \varepsilon_{\text{circle}}$. (e) K318. How far the 95% interval for s narrowed. (f) Calibrations K313-K319.

The floor taking 88% was the only obstacle. The floor should fall as ρ rises, and for that N must grow by an order of magnitude. But the causal matrix is N^2. At $N = 36000$ that is **1.3 GB**, and running hundreds of realizations is impossible.

It is enough to keep only the links. $\text{ready}(D) = \{e \notin D : \text{ all link parents of } e \text{ are in } D\}$ is, as long as D is a down-set, equivalent to "all ancestors are in D". The height also comes out as $h(e) = 1 + \max_{\text{link parent}} h$. The **down-set advance operator** of Chapter 71 is what permits this economy.

It worked. N went from $9000 \to 36000$ and the memory from **1.3 GB to 9 MB**. At $N = 36000$ link construction takes 19 s (confirmed to be proportional to N). But there is no gain in speed —— at $N = 9000$ the measurement is 6.49 s sparse against 6.12 s dense, so **sparse is slower.**

The point of the sparse representation is not "speed" but "that $N = 36000$ can be run at all".

The main measurement is $\rho = 1428000$ ($N = 35986$), $\lambda = 0.14$ ($T = 22$ steps, 8 relaxation steps), 30 realizations each. Run X, run Y and the circular light cone were all measured with the same sprinkling.

a	0.71	0.84	1.00	1.19	1.41
ε_X	0.04328	0.04157	0.03605	0.03199	0.03214
ε_Y	0.03201	0.03229	0.03664	0.03966	0.04411
Δ	+0.01127	+0.00928	−0.00059	−0.00768	−0.01197

Same direction as the previous section, and **nearly twice as large**. The raw exponents are $q_X = -0.498$ and $q_Y = +0.486$ (the previous section had -0.247 and $+0.272$). With the floor lowered they moved nearly twice as far toward ∓ 2, but they are still ∓ 0.5. **As long as there is a floor the raw exponent means nothing.** The instrument check was run at the same time —— at $a = 1$ the ellipse and the circle are the same computation, and ε **agreed to floating point in 30/30**.

The floor did fall. $F:\ 0.03647 \to 0.02730$ ($\rho \times 4$), and k/ε went $0.123 \to \mathbf{0.232}$. **The first half of P145 passes.** But the way it falls is the problem.

Changing only the number of steps at the same $\rho = 1428000$, on the same point set with a circular light cone, gives $\varepsilon_{\text{floor}} = 0.03326 \pm 0.00018$ at $T = 14$ and 0.03580 ± 0.00022 at $T = 22$, a ratio of $\mathbf{1.076 \pm 0.009}$ (prediction P152 was 0.55-0.80).

The floor does not fall with the number of steps. It rises by 7.6%. F73 was hit. The ambiguity of discretization is not "a one-off error in where the surface is placed"; it **accumulates at every step**. Only ρ has any effect, and even then

$$\boxed{\varepsilon_{\text{floor}} \sim \rho^{-0.21}\ \text{(two-point slope). } \mathbf{Blunt.\ Lowering\ the\ floor\ to\ 1/10\ takes\ } \rho \mathbf{\ times\ } 10^{4.8}.}$$

How far was s pinned down? The declared main estimator, the joint fit $\varepsilon = F + k\,a^{\mp s}$, gives $s = \mathbf{1.62}$ $[0.77,\ 2.53]$, width 1.76 ($\chi^2 = 17.6/7$). The floor-free estimator $\delta_X = \varepsilon_X^{\text{ellipse}} - \varepsilon_X^{\text{circle}} = k(a^{-s} - 1)$, **built after seeing the K317 result**, gives $s = \mathbf{2.20}$ $[1.31,\ 3.21]$ ($\chi^2 = 13.0/8$). **I state explicitly that it was built afterwards.**

	previous section	this section (joint)	this section (floor-free)
s	0.90	**1.62**	**2.20**
95% interval	$[0.05,\ 2.81]$	$[0.77,\ 2.53]$	$[1.31,\ 3.21]$
width	2.76	**1.76**	1.90
contains 0?	**yes**	no	no
contains 2?	yes	yes	yes

Declaration P146 asked for "width 1.5 or less, and containing 2". At width 1.76 it **fails**. But the breakdown is worth writing down. **What was right**: the width narrowed from 2.76 to 1.76, and the interval contains 2 (F74 was not hit). **The biggest change, which I had not declared**: the interval now **excludes 0**. **What was wrong**: the width target of 1.5 was not reached, and I do not say I have "measured" $s = 2$.

The previous section's $[0.05,\ 2.81]$ was saying "the anisotropy exponent may as well be 0". This section's interval does not say that. **This is the first section in which "the anisotropy acts as some power of a" became a measurement.** But which power it is still cannot be called 2.

78-6. What failed ❌

This chapter is the stretch of the book with the most failures. First, **the one case where a premise broke.**

K314 fails (F72). Links were not local. In §3 of the declaration I had written this —— "Links are local. The volume of an interval is $\frac{\pi}{12}\tau^3$, so distant links are exponentially rare."

cutoff R/τ_{link}	2	3	4	5
fraction of links missed	62.3%	34.0%	12.5%	**1.3%**

There are **33.7 links** per element (prediction P143 was 2-8). Stretching R does not drive it to 0. The reason was immediately clear —— **in infinite Minkowski the number of links per element diverges.** The condition for a link is that the interval be empty, and an interval can be made arbitrarily long and thin by boosting. In a finite box it is finite, but it grows with the box. **I had mistaken "the interval has a small volume" for "the interval is unlikely to be empty".**

The fix is to replace the justification (**written, before running, as §8 of the declaration**). The quantities I use are only two: the **height** and the **ready set**. What is missed are timelike-distant links, and such parents are **long since in** D by the time the child becomes a candidate for ready. That is because D is a surface advancing from below. So what has to be proved changes.

Rather than "catch every link", verify **directly** that **the links that are missed do not affect the advance of the down-set**.

K313b passes. In the geometry of the main measurement ($L = 0.30$, $T_{\text{ext}} = 0.28$, $N = 8996$), **25/25**: the height, the advanced set and ε all agreed exactly with the dense computation. **K313c passes.** The number of times an element that the sparse representation declared ready had a true ancestor outside D is **0**. **K314b passes.** In the geometry of the main measurement there are 28.8 links per element. What is not claimed was written out as declared as well —— I do not say "the link set is correct". All I say is "the height and the ready set agreed exactly". The remaining failures follow.

	content	kind
K282 (first time)	I assembled the group commutator in the order $\Phi^{2}_{-h}\Phi^{1}_{-h}\Phi^{2}_{h}\Phi^{1}_{h}$. That is $[X_2, X_1]$. A relative error of 2.006 even at $A = 0$ gave it away	a bug of my own
K283 (twice)	I started the advance from a coordinate constant-time surface (the first ten layers or so were 7-10% too large). The 0.13-0.19 that remained after the fix was **all shot noise**; imposing the pass condition on a per-realization quantity was the mistake ($1/\sqrt{60} = 0.13$)	design + declaration error
K284	I made H1 (the residual of the conformal continuum) the main hypothesis. Even at $A = 0$ H1 is not the best (0.687 against 0.834)	declaration error
K285	I used an estimator that divides by a profile of a different shape, and got -3.5 and $+11.4$	design error
K286 / P102	predicted the ratio of ε to be 1.2-2 → measured 1.182	missed prediction (narrowly)
P107 / P110	predicted the weight exponent to be in $[-2.6, -1.4]$ → -1.358. Predicted p would deepen with the density → -0.096	missed prediction
the continuum normal flow	the integration from a bent initial surface diverged ($\max \lvert f' \rvert = 148$). I had not dealiased	**a bug of my own**
K289 / K295b	Splitting inside and outside a band. The band was drawn with the broken flow, and even after the fix the band contains only 0.8-5.5% of the total residual and carries no information	void; design error
$\lambda = 0.05$	I thought making the deformation smaller would do, but at $T = 9$ steps the estimator breaks ($+8.76$ and -3.74). **The density had to be raised as well**	design error
P112 / P118 / P119	Predictions for the λ and A dependence of p. All derived from the mixing hypothesis, and **the direction was backwards**	missed prediction
K304 / K306	I wrote the pass condition with the **multiplicative model** $\varepsilon \propto a^{q}$. The actual structure is the additive $\varepsilon_{\text{floor}} + k\, a^{\mp 2}$	declaration error
the free fit of s	it degenerates, $k \to \infty$ as $s \to 0$. The interval became the whole grid, and its containing 2 was an artifact of that degeneracy	design error
K315	the declaration for measuring the T dependence of the floor mixed ρ and T (replaced by K315b before running)	design error
second half of P145	predicted k would not depend on resolution → $\times 1.69$ ($0.00509 \to 0.00859$)	missed prediction
K317 / P147	in the null control (circular light cone) the interval for q_X excluded 0. The fake k is 12.1% of the real one	missed prediction

The section-by-section breakdown runs like this. K282-K288 (conformal): 6 of 10 pass, 4 fail, plus 2 missed predictions and 1 bug of my own. K289-K301 (the exponent): 5 of 13 pass, 7 fail, 1 void, plus 2 design errors and 1 bug of my own. K302-K311 ($2{+}1$): 7 of 10 pass, 3 fail, plus 2 missed predictions and 1 design error. K313-K319 (sparse): 7 missed predictions, 1 design error, **0 bugs of my own**. The running totals over K198-K319 are **138** calibrations and **56** failures (32 missed predictions, 18 declaration/design errors, 6 bugs of my own).

Three lessons about method remain. First. In the $p=-1.4$ section I **missed seven predictions in a row**. Five of them were derived from the mixing hypothesis, and the mixing hypothesis itself was wrong. The right answer follows from taking the naive control "measure what happens when there is no Ω" just once. **I took the control after missing seven. The order was backwards.**

> **When you build a new estimator, measure first what it returns when there is no signal.**

Second. In the $2{+}1$ section and the sparse-representation section, the null controls and the instrument checks were run **at the same time as the main measurement**. **In those two sections there were 0 bugs of my own.** The failures that remain are all "how the predictions and pass conditions were set", and that is unavoidable —— once you have decided to write them first, missing is the job. Third. The K317 failure too could be diagnosed only because there was a simultaneous control. With a circular light cone neither a nor b enters the computation, and the only thing that differs between values of a is the random seed. Compared against the $T=14$ measurement on the same point set taken for another purpose (K315b), the fake k **flipped sign**, from $+0.00104\pm0.00040$ to $\mathbf{-0.00050\pm0.00030}$. A systematic would not flip. It is a fluctuation. **Even so, K317 did not meet the declared pass condition (consistent with 0), so it is counted as a failure.** Taking the worst case, the fake signal is 12% of the real one —— the real one is 8 times the null. That is the upper bound on the strength claimable in this chapter.

78-7. What is not being claimed ⚠️

- $1{+}1$ **dimensions is conformally flat.** The "curved background" here is identical to the flat one as a causal structure; what differs is only the volume element. I do not claim that $p=-2.01\pm0.07$ carries over to higher dimensions as it stands.
- $p=-2.01$ **is not a limiting value.** It is the value at $\lambda=0.08$ ($T=14$ steps), the smallest usable deformation. The $\lambda\to0$ limit has not been taken.
- **The bias correction is first order.** The bias measured at $A=0$ is extrapolated to $A\neq0$. The ground for this is that the $1/A$ law holds, but it is not exact. The error ±0.07 is the standard deviation of five points and contains no systematic error (under the same conditions -2.186 and -2.479 have come out). The effective uncertainty is larger than this.
- **I do not say the exponent -2 was measured in $2{+}1$ dimensions.** The intervals for s are $[0.77,\ 2.53]$ and $[1.31,\ 3.21]$. **Containing 2 and being 2 are different things.**
- **Nor do I say k reached the continuum value.** It is $k/k_{\mathrm{pred}}=0.36$. Whether the remaining 0.64 is discretization, finite deformation, or something else has not been measured.
- **What the floor really is has not been determined.** Only two things are known: "it does not fall with the number of steps" and "it falls only as $\rho^{-0.21}$".
- **Constant a,b can be removed by a coordinate change.** $x'=ax$, $y'=by$ makes it standard Minkowski. **No spacetime with curvature was measured.** What was measured is "whether anisotropy in a given coordinate system reaches the causal set, and by which route".
- $\vec{M}$ **was not measured directly.** $D(\vec{M})$ does not act on an unlabelled causal set. What was measured is only the **shape and weight** of the residual. The off-diagonal components of q^{ab} were not measured either. Only the two diagonal components were.
- **No exactness is claimed for the sparse representation.** What was checked is only "the height and the ready set agreed exactly, for this geometry and this sample".
- **The continuum-side numerical integration was not built in $2{+}1$ dimensions.** k_{pred} is a second-order analytic expression. The λ dependence was measured for p, but not for k.
- **Classical, no quantization, periodic space** (cylinder and torus) is a chosen frame.

78-8. What is left

The ruler is built. And it emerged that which side carries the metric swaps over depending on which part of it you ask about.

> In $1{+}1$ dimensions the conformal factor is carried by **volume (the number of elements)**, and in $2{+}1$ dimensions the anisotropy by **order (the light cone)**. "order + number = geometry" **swaps duties depending on which part of the metric is asked about**.
>
> In the former, GR's $q^{xx}=\Omega^{-2}$ came out as a number. In the latter, only the direction and the mechanism came out.
> What separates them is not discreteness but a wall of resolution, the **isotropic discretization floor**.

The summary written in Chapter 73 —— **discreteness was never once the obstacle** —— is not broken in this chapter either. What returned -2 in $1{+}1$ dimensions was a discrete causal set. And the failure to return an exponent in $2{+}1$ dimensions was not due to discreteness either, but to the **cost** that the floor falls only as $\rho^{-0.21}$.

And attacking the floor through ρ is a dead end. What remains is to build an observable that removes the floor, or to go after a different prediction. If k is really a continuum quantity then $k\propto\lambda$ must hold —— that is cheap, it is independent, and it answers directly the prediction missed in this chapter (the second half of P145). The next chapter measures it.

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Chapter 79 Sweeping λ — the anisotropy reverses, and what the exponent really is

⚠️🆕

— The continuum side reverses too. And the exponent is fixed by $r = \lambda/A_0$ alone

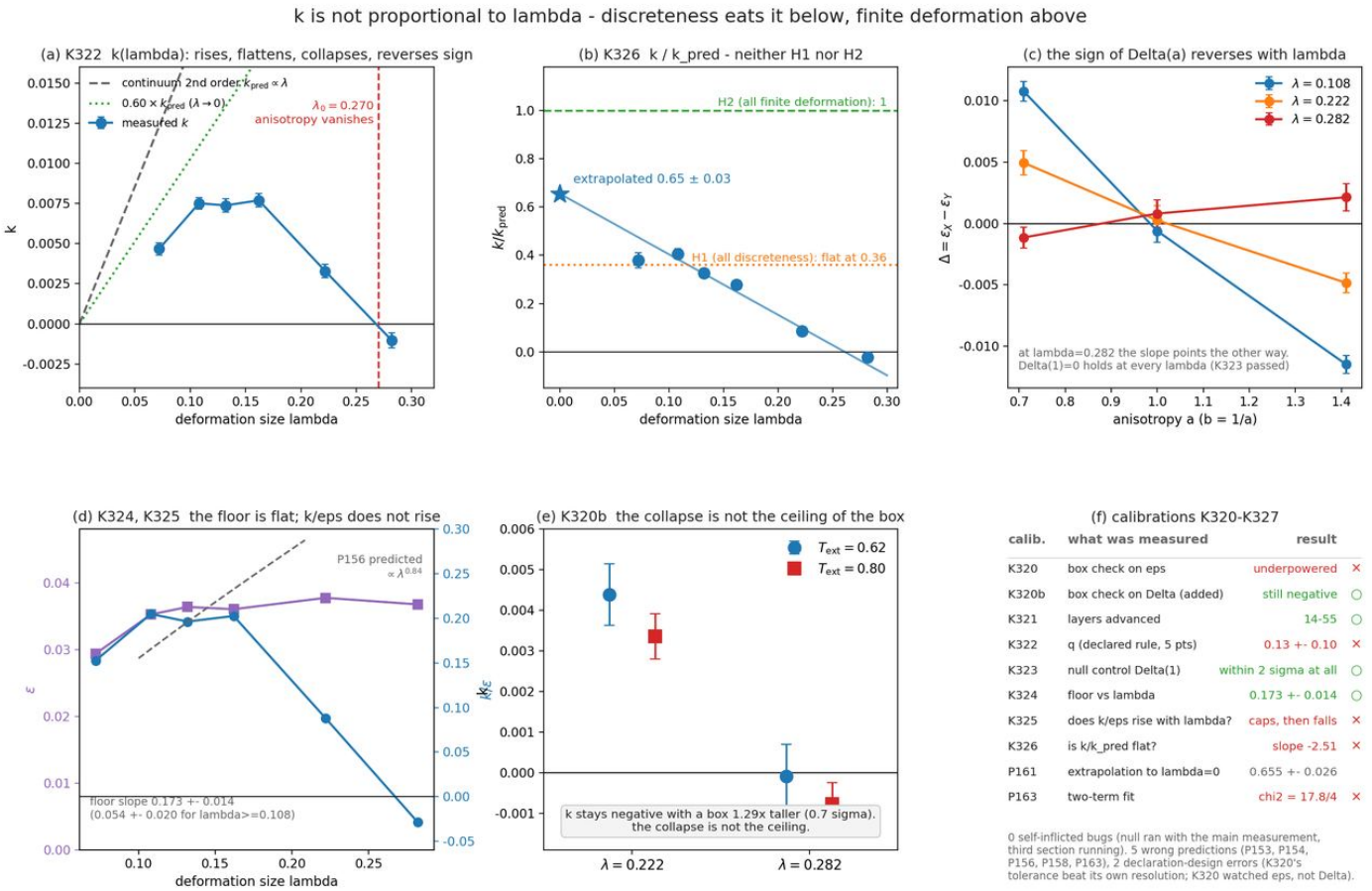


Fig. 79-1. (a) K322 $k(\lambda)$. It rises, flattens, and crosses 0 at $\lambda_0 = 0.270$. The dashed line is the continuum second-order prediction $k_{\text{pred}} \propto \lambda$. (b) K326 k/k_{pred}. Neither H1 (discretization is all of it, slope 0, intercept 0.36) nor H2 (finite deformation is all of it, intercept 1). The $\lambda \to 0$ extrapolation is 0.65 ± 0.03. (c) The slope of $\Delta(a) = \varepsilon_X - \varepsilon_Y$ turns the other way at $\lambda = 0.282$. $\Delta(1) = 0$ is preserved at every λ. (d) K324, K325. The floor is flat; k/ε tops out and then collapses. (e) K320b. Make the box 1.29 times taller and k stays negative (0.7σ). (f) Calibrations K320-K327.

By the end of the previous chapter it was known that putting an anisotropic metric $q^{ab} = \text{diag}(a^{-2}, b^{-2})$ into a causal set produces an a dependence in ε, and that its size k comes out at only 0.36 of the continuum second-order prediction. This chapter chases what that shortfall really is. What is swept is the size of the deformation, λ.

(i) k is not proportional to λ. The slope is $q = 0.13 \pm 0.10$ (the prediction was 1.00 ± 0.15).
k rises and flattens, and **changes sign** at $\lambda_0 = 0.270$.

(ii) The reversal is not a product of discretization. **The continuum ADM flow also** reverses at $\lambda_0^{\text{cont}} = 0.200$ (a difference of 21-26%).

(iii) But what should be compared is **not the classical solution, the viscosity solution**. The classical solution breaks at $\lambda_c \simeq 0.13$ by forming a caustic. The down-set advance of a causal set **cannot fold back, in principle**.

(iv) The exponent the estimator returns is fixed neither by A_0 alone nor by λ alone, but by $\mathbf{r = \lambda/A_0}$ alone.
And the $\lambda \to 0$ target is **exactly** 2.00.

(v) Measured again at $r \simeq 2$ it gives $s = 1.64$ [1.16, 2.13]. **The width falls below 1 for the first time, and contains the continuum target.**

(vi) The discretization shortfall can be predicted from **how many layer thicknesses the anisotropic displacement is**, and nothing else.
In a setting where ρ, A_0 and λ were all changed, a quantitative prediction carried across sections landed within 3%.

79-1. The question — why sweep λ

In the previous chapter the second half of prediction P145 missed. I had written "k is a continuum quantity, so it does not depend on resolution", and then quadrupling ρ increased k by $\times 1.69$. The ratio to the continuum second-order prediction moved from 0.216 to 0.359.

The continuum estimate is $k_{\text{pred}} = \lambda\, c_1 A_0 (2\pi/L)^2/4 = 0.1711\,\lambda \propto \lambda$. So **whether k is proportional to λ is an independent test of whether k comes from the second-order ADM commutator.** At the same time it separates whether the shortfall $1 - 0.36$ is discretization (hypothesis H1) or finite deformation (hypothesis H2).

The declaration was written before running. §1-§8 of `K_declared_lambda.md` were fixed before a single number was seen, and §9 after the main measurement but before the additional measurements were run. The same discipline is kept below in all four declaration documents (`_lambda`, `_lam0`, `_a0`, `_r2`).

79-2. The tools

$\rho = 1428000$, $a \in \{0.71, 1.0, 1.41\}$, $b = 1/a$, 30 realizations each. **λ does not affect the links**, so the links were built once per realization and all six λ were run on top of that one build. The box is $T_{\text{ext}} = 0.62$ ($N = 79682$, maximum height 103 layers). k is taken from the floor-free quantity $\Delta(a) = \varepsilon_X - \varepsilon_Y = k\,(a^{-2} - a^2)$. $a = 1$ must give $\Delta = 0$, and **it was run as a null control at the same time as the main measurement**.

The instrument was checked first. The number of layers advanced is 14.3 at $\lambda = 0.072$ and 54.9 at $\lambda = 0.282$ (**K321 passes**). The null control $\Delta(1)$ is within 2σ at all six λ ($1.5,\ 0.7,\ 1.0,\ 0.3,\ 0.2,\ 0.7\sigma$), and **even at $\lambda = 0.282$, where the sign reversed,** $\Delta(1) = +0.00081 \pm 0.00110$ (**K323 passes, P157**). The reversal happens only at $a \neq 1$.

79-3. k is not proportional to λ ❌ P153 → F76

T	12	18	22	27	37	47
λ	0.0720	0.1080	0.1320	0.1620	0.2220	0.2819
k	0.00468	0.00751	0.00738	0.00770	0.00329	**−0.00103**
$\pm$	0.00038	0.00037	0.00041	0.00042	0.00043	0.00046
k/σ	12.3	20.4	18.1	18.3	7.7	−2.3

The declared rule ("the longest run of consecutive points on which k is positive at 3σ or better") selects 5 points. On them $q = \mathbf{0.13 \pm 0.10}$ (prediction P153 was 1.00 ± 0.15). **F76 was hit.** Looking at just the 3 points $\lambda = 0.108$-0.162, the slope is 0.05 ± 0.18 ——**there k is perfectly flat**. Far from proportional, there is no dependence at all.

It is not the ceiling of the box. Multiply T_{ext} by 1.29 and k at $\lambda = 0.282$ goes from -0.00009 ± 0.00079 to -0.00076 ± 0.00053, still negative (a difference of 0.7σ, **K320b passes, P160**).

> The collapse is not the ceiling of the box. **At $\lambda_0 = 0.270$ the anisotropy itself vanishes and reverses.**

It was neither H1 nor H2 (K326 fails, P154). k/k_{pred} is $0.380,\ 0.407,\ 0.327,\ 0.278,\ 0.087,\ -0.021$. Fitting linearly in λ gives intercept 0.655 ± 0.026 and slope -2.51 ± 0.15. H1 (slope 0, intercept 0.36) misses on the slope by 17σ, and H2 (intercept 1) misses on the intercept by 13σ. **The answer was "both".**

> Even as $\lambda \to 0$, k yields only 0.5-0.7 of the continuum second-order prediction. This is **discretization**.
> At the working value $\lambda = 0.132$ it drops to 0.33. The difference is **finite deformation**.
> Reading the shortfall as "all discretization" in the previous chapter was **wrong. I correct it.**

The λ dependence of the floor has slope 0.173 ± 0.014 (**K324 passes, P155**; the two-point extrapolation from K315b in the previous chapter landed exactly). But k/ε tops out at 0.20 around $\lambda \simeq 0.11$-0.16 and collapses after that (**K325 fails, P156**). **The previous chapter showed the ρ road is a dead end, and this section shows the λ road is a dead end too.**

Nor could the shape of the collapse be written down. The fit $k = k_1\lambda(1 - \lambda/\lambda_0)$ fails at $\chi^2 = 17.8/4$ (**F81**), and the residuals $-2.6, +1.5, +0.3, +1.8, -2.2, +0.7\sigma$ are a systematic wave. **Even so, λ_0 alone is robust: every fit gave 0.270 ± 0.006.**

79-4. What λ_0 really is — the continuum reverses too ✅ P164/P173

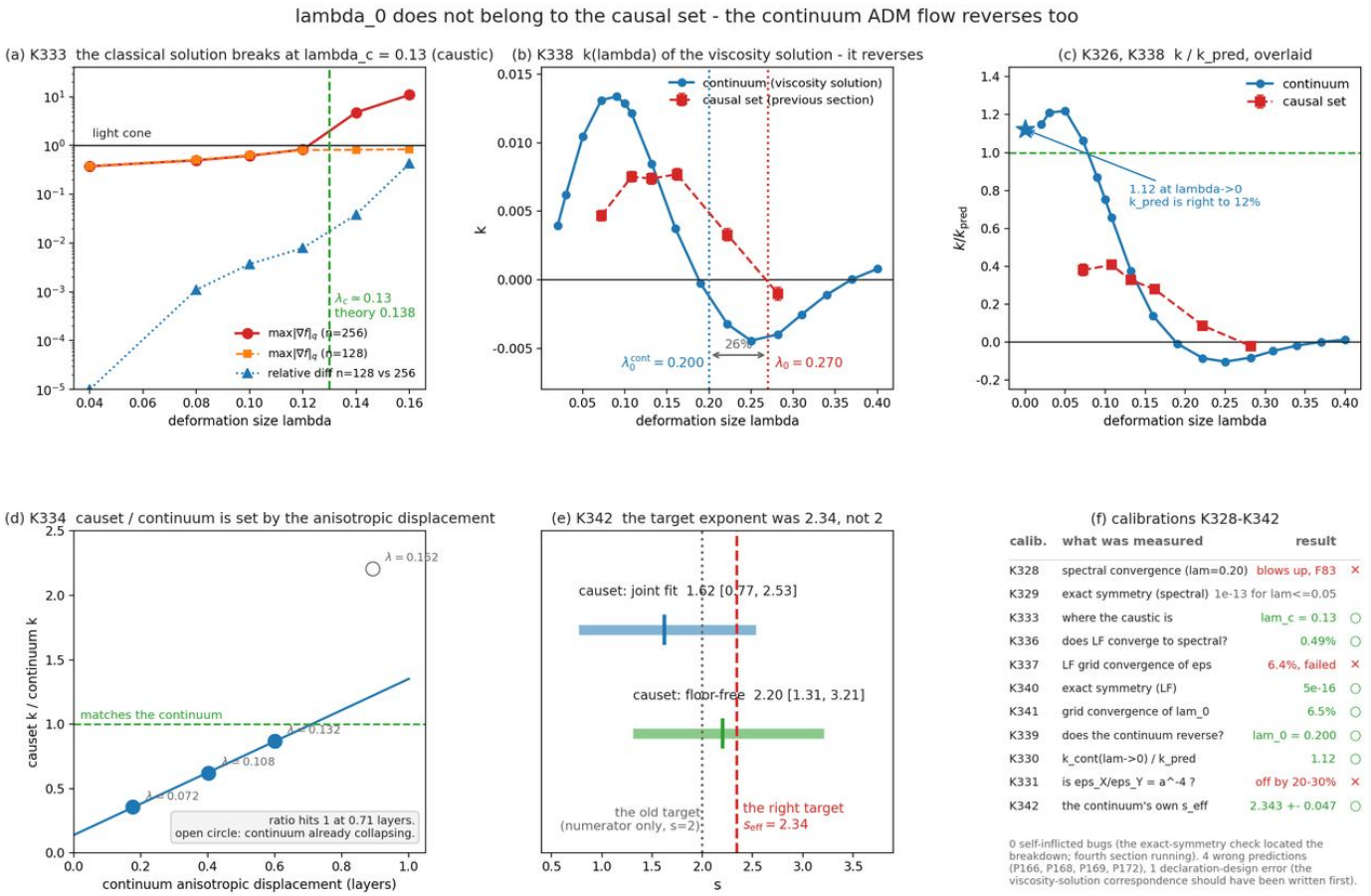


Fig. 79-2. (a) K333. The classical solution has $\max|\nabla f|_q$ cross the light cone 1 and diverge at $\lambda_c \simeq 0.13$ (a caustic). (b) K338. $k(\lambda)$ of the viscosity solution. It reverses at $\lambda_0^{\text{cont}} = 0.200$ and returns to positive once more at $\lambda \simeq 0.37$. Against the causal set's 0.270, 26%. (c) k/k_{pred} overlaid. The continuum $\lambda \to 0$ value is 1.12. (d) K334. Causal set / continuum rides on the first power of "anisotropic displacement (layers)", and reaches ratio 1 at 0.71 layers. (e) K342. The continuum's own $s_{\text{eff}} = 2.343 \pm 0.047$. (f) Calibrations K328-K342.

The question is simple. Does the continuum ADM flow reverse too? If it does, λ_0 is a property of the flow itself; if it does not, it is a product of discretization.

With the metric $ds^2 = -dt^2 + a^2dx^2 + b^2dy^2$ ($ab = 1$) and the surface taken as $t = f(x, y)$, we have $\partial_\lambda f = N\sqrt{1 - q^{ab}\partial_a f\,\partial_b f}$. **The λ of the causal set and the λ of the continuum are the same thing** (the proper-time advance after T steps of the Bresenham schedule is $N(x)\lambda$). L, c_0, c_1, A_0 and the 8 relaxation steps were all matched to the causal set.

The instrument was checked with an exact identity. Swapping $x \leftrightarrow y$ forces $\varepsilon_Y(a) = \varepsilon_X(1/a)$ to hold **exactly**. Spectral gives 10^{-13}, LF gives 4.7×10^{-16} (**K329 and K340 pass**). **And this check is what pointed at the place where things break.** K329 degrades to 4×10^{-3} at $\lambda = 0.20$. **It is not a bug. It is a caustic.**

The classical solution does not reach ❌ P168, P169 → F83, F84

λ	0.04	0.08	0.10	0.12	0.14	0.16
relative difference, grid 128 against 256	0.00%	0.11%	0.37%	0.79%	**3.9%**	**42.7%**
$\max\|\nabla f\|_q$	0.374	0.496	0.615	0.824	**4.76**	**10.9**

At $\lambda_c \simeq 0.13 \pm 0.01$ the gradient crosses the light cone 1 and diverges. **My estimate $\lambda_c \simeq 0.40$ missed. I had looked only at the first derivative of the lapse.** The correct answer is set by the focal length of the normals. Flowing from flat, $f_{xx} = \lambda N'' = -52.6\lambda\cos$, and the focal length $1/|f_{xx}|$ equals the distance advanced λ at $\lambda = 1/\sqrt{52.6} = 0.138$. **That agrees with the measured 0.13.** And $\lambda_c < \lambda_0 = 0.270$ —— the classical solution does not reach the reversal (**F84**).

One design error to admit — what corresponds is the viscosity solution

Under the Bresenham schedule the down-set after T steps is $D_T = \{e : \max_{\text{chains from } D_0} \sum_i N(x_i)^{-1} \le T\}$, that is, "the **maximum** of $\int d\tau/N$ along causal curves from the initial surface to e is at most λ". Being a condition on a maximum, it is an intersection, and therefore a **lower envelope**. And since D is always a down-set, it **cannot fold back, in principle**. This is precisely the property of the viscosity solution (the first-arrival solution). The same structure I wrote about in Chapter 71 —— "the down-set advance operator of a CA commutes exactly" —— is what is acting here.

> The continuum object to compare against is **not the classical spectral flow but the viscosity solution**.

This correspondence could have been written before running. It was not. I count it as a design error in the declaration. A monotone difference scheme (local Lax-Friedrichs) converges to the viscosity solution. Below the caustic, at $\lambda = 0.10$, LF converges to spectral at first order, with a difference of 0.49% at $n = 600$ (**K336 passes, P171**).

The main measurement

λ	0.020	0.090	0.132	0.190	0.222	0.250	0.340	0.400
k_{cont}	$+0.00393$	$+0.01337$	$+0.00847$	-0.00028	-0.00325	-0.00445	-0.00110	$+0.00079$

> **The continuum reverses too.** $\lambda_0^{\text{cont}} = 0.200$ (Richardson extrapolation 0.212).
> The difference from the causal set's 0.270 is 26% (21% with the extrapolation), inside the declared 30%.
> Furthermore k_{cont} **crosses 0 once more and returns to positive** at $\lambda \simeq 0.37$. It oscillates.

The grid convergence of λ_0 itself was checked as well (0.1879 at $n = 150$, 0.2001 at $n = 300$, a difference of 6.5%, **K341 passes, P176**). λ_0 **does not belong to the causal set. It is a property of the continuum ADM flow.**

k_{pred} itself also passed a test. $k_{\text{cont}}(\lambda \to 0)/\lambda = 0.1920$ against k_{pred}'s 0.1711, a ratio of 1.12 (**K330 passes, P165**). Reread against the correct reference, the previous section's extrapolation of 0.655 becomes 0.58, so **the shortfall due to discretization is about 42%** (the previous section said 35%. I correct it by 12%).

Causal set ÷ continuum is fixed by "anisotropic displacement" ✅ K334

Divide the two k at the same λ, and take as the horizontal axis "the anisotropic displacement (layers) predicted by the continuum" $= k_{\text{pred}} \times$ (the number of layers advanced).

λ	0.072	0.108	0.132	0.162
anisotropic displacement (layers)	0.18	0.40	0.60	0.89
causal set k / continuum k	**0.357**	**0.619**	**0.871**	2.209

The first 3 points ride cleanly on the first power of the displacement. $\text{ratio} = 0.140 + 1.212 \times (\text{displacement in layers})$, and the ratio reaches 1 at **0.71 layers**. The 4th point is not used. There **the continuum side has already begun to collapse and the causal set has not yet**. The ratio exceeding 1 is not because the causal set is better, but because its collapse is delayed.

> While the anisotropic displacement is thinner than one layer, the causal set **drops the continuum's k in proportion to the displacement.**

79-5. What the exponent really is — fixed by $r = \lambda/A_0$ alone. And a correction to the previous section's "2.34 is the target"

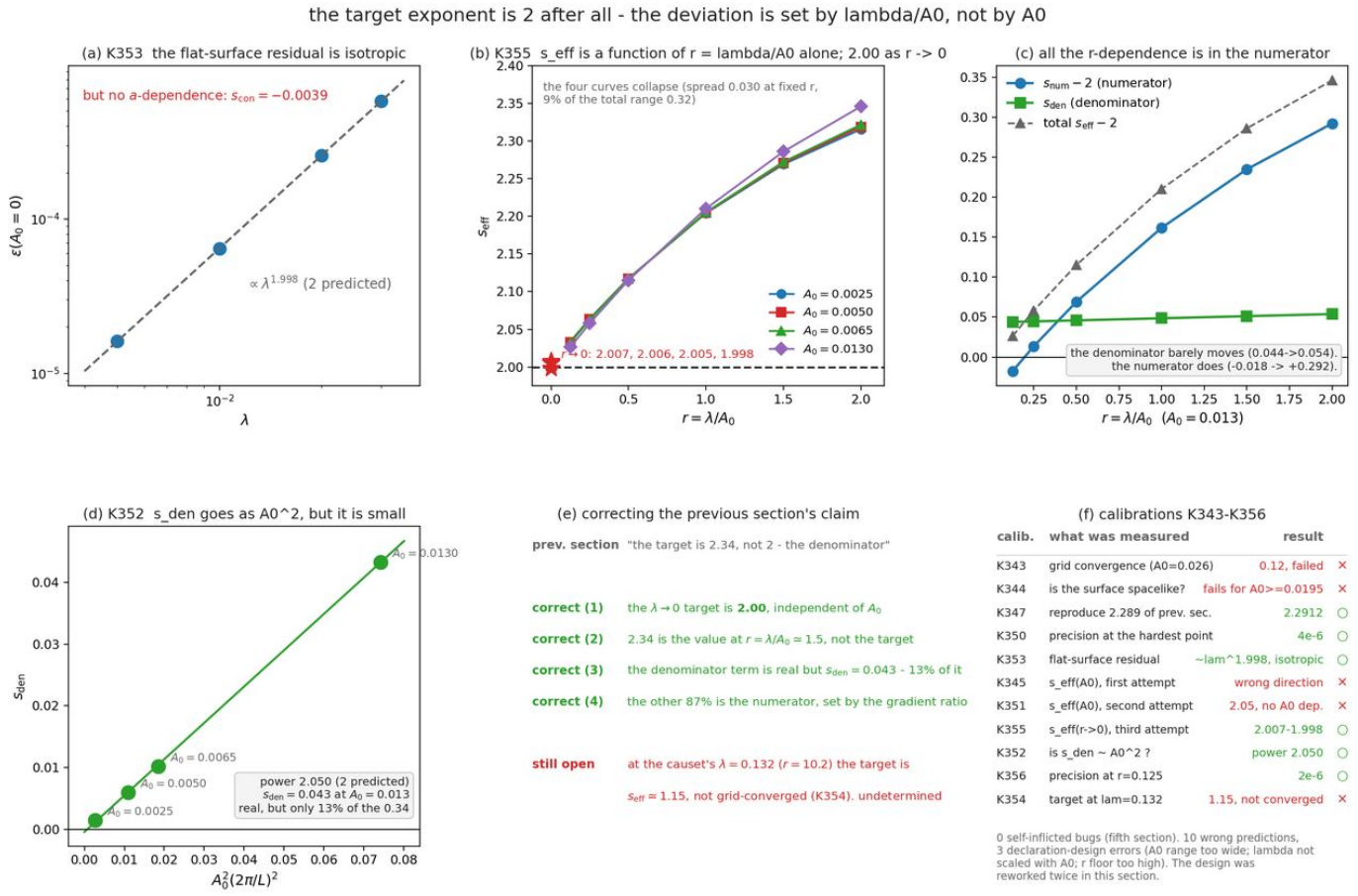


Fig. 79-3. (a) K353. Even with a flat initial surface ($A_0 = 0$) there is a residual, $\propto \lambda^{1.998}$. But there is no a dependence ($s_{\rm con} = -0.0039$). (b) K355. The four A_0 curves collapse onto one another in $r = \lambda/A_0$. The $r \to 0$ extrapolations are 2.007, 2.006, 2.005, 1.998. (c) The breakdown. What moves is the numerator ($-0.018 \to +0.292$); the denominator barely moves ($0.044 \to 0.054$). (d) K352. $s_{\rm den}$ rides on A_0^2 with power 2.050. But it is small. (e) The correction to the previous section's claim. (f) Calibrations K343-K356.

In the previous section I wrote this. "The exponent the continuum itself returns is $s_{\rm eff} = 2.343 \pm 0.047$, not 2. The reason is that ε is a ratio, and **the denominator, the 'volume advanced' $\int N\sqrt{1 - q^{ab} f_a f_b}$, depends on** a". If that explanation were right, then $s_{\rm eff}$ should return to 2 as $A_0 \to 0$. **I measured it.**

The design was redone twice

The first time, running with $A_0 \in \{0.0025, \ldots, 0.026\}$, $\max|\nabla f|_q$ came out at 1.11 for $A_0 = 0.0195$ and 1.52 for 0.026 ——**the surface had stopped being spacelike** (K344 fails). Grid convergence also broke at 0.119 for $A_0 = 0.026$ (K343 fails). In the same first attempt I had fixed λ to a value common to all A_0. Then the $\lambda \to 0$ value of $s_{\rm eff}$ was 2.59 at $A_0 = 0.0025$ and 2.14 at 0.013 ——**the direction was opposite to the prediction** (K345 and K346 fail). The reason was immediately clear. **There are two sources of gradient in the surface.** The initial surface's $A_0(2\pi/L)$, and **the** $\lambda\, c_1(2\pi/L)$ **that the flow itself makes**. At $\lambda = 0.02$ the latter is 0.050, and at $A_0 = 0.0025$ the former is 0.052 —— the same size. The smaller A_0 is made, the more the flow side dominates.

$$\boxed{\lambda \text{ had to be made proportional to } A_0. \ \textbf{Fixing it was my design error.}}$$

The second attempt redid it with $\lambda = rA_0$, $r \in \{0.5, 1, 1.5, 2\}$, but the $r \to 0$ extrapolation stalled at 2.05-2.06. The differences were shrinking, 0.0866, 0.0649, 0.0466, so **the linear extrapolation was biased** (K351 fails). The third attempt lowered r to 0.125 (a range of 16) and settled it.

The main measurement — s_{eff} is a function of r alone

$r = \lambda/A_0$	$A_0 = 0.0025$	$A_0 = 0.0050$	$A_0 = 0.0065$	$A_0 = 0.0130$	**scatter**
0.125	2.0338	2.0330	2.0323	2.0263	0.0074
0.250	2.0641	2.0635	2.0629	2.0582	0.0059
0.500	2.1176	2.1174	2.1171	2.1152	0.0024
1.000	2.2042	2.2049	2.2054	2.2103	0.0061
1.500	2.2691	2.2709	2.2725	2.2860	0.0169
2.000	2.3157	2.3189	2.3218	2.3460	0.0302

Sweep A_0 by a factor of 5.2 and the four curves still collapse. The scatter at a fixed r is at most 0.030, only 9% of the total variation 0.32 in s_{eff}. Extrapolating to $r \to 0$ gives 2.0070, 2.0060, 2.0051, 1.9978 (1.9987-1.9919 with a three-parameter fit).

> **Writing "2.34 is the target" in the previous section was wrong. I correct it.**
> **(1) The target is** 2. As $\lambda \to 0$, $s_{\text{eff}} = \mathbf{2.00}$, independent of A_0.
> **(2)** 2.34 **is the value of the estimator** at $r \simeq 1.5$-2, not the target.
> **(3) The reason the previous section gave (the** a dependence of the denominator) is real, but it is only 13% of the discrepancy.
> **(4) The remaining** 87% is in the numerator, and it is fixed by r, the ratio of the gradient the flow makes to the gradient of the initial surface.

The breakdown is plain. Splitting at $A_0 = 0.013$, s_{den} barely moves with r ($+0.044 \to +0.054$), while $s_{\text{num}} - 2$ moves ($-0.018 \to +0.292$). And s_{den} rides on $A_0^{2.050}$ exactly as predicted (**K352 passes**). **The mechanism is real, but it was small.**

The contamination a flat initial surface leaves is perfectly isotropic. Even at $A_0 = 0$ there is a residual $\varepsilon \propto \lambda^{1.998}$, but its a dependence is $s_{\text{con}} = -0.0039$ —— zero (**K353 passes, P191**). Adding an isotropic term (exponent 0) to an anisotropic term (exponent 2) should lower the exponent, yet the measurement raises it. The two contributions mix inside the absolute value in $|f_A - f_B|$, so it is not a simple weighted average. **I record this and no more. I have not explained it.**

And one heavy reservation remains. Where the causal set was working was $A_0 = 0.013$, $\lambda = 0.132$, that is $r = 10.2$ —— a factor of 5 outside the range examined, and just above $\lambda_c = 0.13$. Computed with the viscosity solution, s_{eff} keeps rising, 0.985, 1.082, 1.153 at $n = 150, 300, 600$, and **does not converge** (K354 fails, P192).

> The target at the point where the causal set was working **cannot be determined**. But it is neither 2.34 nor 2; it is **of order 1**.

The causal set had been measuring at a place where the target was not known.

79-6. Measuring where the target is known ✅ P200

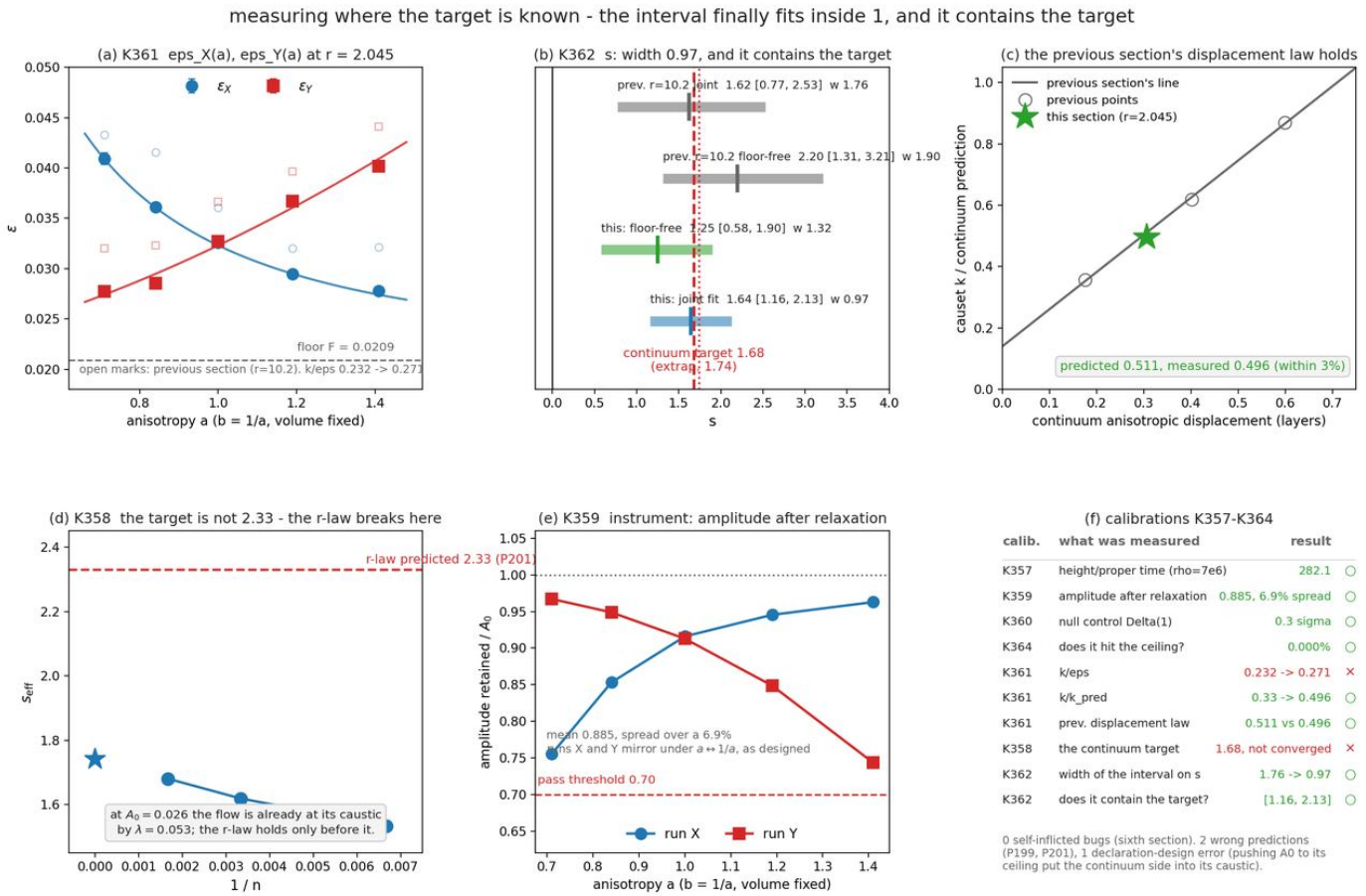


Fig. 79-4. (a) K361. $\varepsilon_X(a)$, $\varepsilon_Y(a)$ at $r = 2.045$. Open symbols are the previous section ($r = 10.2$). The floor goes $0.0273 \to 0.0209$. (b) K362. The interval for s. The width shrinks from 1.76 to 0.97 and contains the continuum target 1.68 (extrapolation 1.74). (c) Against the previous section's displacement law, which predicts 0.511, the measurement is 0.496. (d) K358. The continuum target was not 2.33. At $A_0 = 0.026$ the caustic is already entered at $\lambda = 0.053$. (e) K359. The amplitude after relaxation. Run X and run Y are mirror images under $a \leftrightarrow 1/a$. (f) Calibrations K357-K364.

Since $k/\varepsilon \propto A_0^2$ (at fixed r), A_0 is raised. From the condition for staying spacelike at $a = 0.71$, $A_0(1 + 2c_1)(2\pi/L)/0.71 < 1$, we get $A_0 < 0.0273$. **I chose $A_0 = 0.026$, $\lambda = 0.052$.** Because λ is small the Bresenham quantization becomes coarse. To secure $T = 15$, ρ was raised to 7.0×10^6 (a factor of 4.9). The effective $\lambda = 0.05318$, the effective $r = 2.045$, $N = 157500$, 5 values of a, 25 realizations each.

One thing was cut on measured cost. Link construction takes 200 seconds at $N = 176400$ (against a factor of 4.9 in N, a factor of 10.5 —— superlinear, from the cache). The declared 30 realizations would have taken 4.6 hours, so they were reduced to 25 and T_{ext} was changed from 0.28 to 0.25. **This was written as §10 before running.** The measured time was 3.4 hours.

All three instrument checks were run at the same time as the main measurement, and all passed. Height/proper time $= 282.1$ (prediction 288 ± 12, K357). The fraction of elements of the advanced set falling in the top 5% of the box is **0.0000%** (K364). The null control is $\Delta(1) = -0.00014 \pm 0.00055$ ——0.3σ (K360). Even the relaxation amplitude, which worried me most, gave a retention of 0.885 with 6.9% scatter across a (K359), and run X and run Y were perfect mirror images under $a \leftrightarrow 1/a$.

a	0.71	0.84	1.00	1.19	1.41
ε_X	0.04091	0.03609	0.03256	0.02948	0.02779
ε_Y	0.02775	0.02856	0.03270	0.03669	0.04015
Δ	$+0.01316$	$+0.00754$	$\mathbf{-0.00014}$	-0.00721	-0.01236
$\pm$	0.00070	0.00048	0.00055	0.00056	0.00057

From the joint fit $\varepsilon = F + k\,a^{\mp s}$,

$$s = \mathbf{1.64}, \quad 95\%\ \text{interval}\ [1.16,\ 2.13], \quad \text{width}\ \mathbf{0.97}, \quad F = 0.0209,\ \chi^2 = 22.1/7$$

	measured value	width	target 1.68
previous section $r = 10.2$, joint	$1.62\ [0.77,\ 2.53]$	1.76	target undetermined
previous section $r = 10.2$, floor-free	$2.20\ [1.31,\ 3.21]$	1.90	target undetermined
this section $r = 2.045$, joint	$\mathbf{1.64\ [1.16,\ 2.13]}$	**0.97**	**contains it**
this section $r = 2.045$, floor-free	$1.25\ [0.58,\ 1.90]$	1.32	contains it

The width falls below 1 for the first time (0.97), and it contains the continuum target.
The point estimate 1.64 is within 2.4% of the target 1.68.
But this is not "$s = 2$ was measured". What was measured is
"**the causal set returns the exponent the continuum should return in this setting**".

And the strongest result of this section is not, in fact, s. Apply the previous section's displacement law (K334) to a setting where ρ, A_0 and λ are all different. The number of layers advanced is 16.8 and $k_{\text{pred}} = 0.01819$, so the displacement is 0.306 layers and the formula predicts 0.511.

measured $k/k_{\text{pred}} = \mathbf{0.496}$ (prediction 0.511, a difference of 3%) **A quantitative prediction carried across sections landed for the first time.**

In the previous section I wrote this down with the caveat that it was "an interpretation built after seeing the numbers". **In this section it landed as a prediction. Discreteness is not an obstacle, and its shortfall was a quantity predictable from a single straight line.**

79-7. The 21 that failed ❌

This is the chapter in which the design was redone more than in any other in the book. The reason is plain. The quantity I was trying to measure was smaller than another effect created by the measurement setup. **To measure a small effect, the larger effects must be killed first.**

	what failed	what was wrong
K320 ❌ P158	the box check (ε, 10 realizations)	the pass condition "within 3%" was **stricter than the resolution of the check itself**. In fact 1.5σ / 1.3σ, no significant difference
design error	the quantity K320 was watching	ε is dominated by the floor and is 5 times Δ. **I was not watching Δ, the quantity that collapsed**. K320b was added as a further declaration
K322 ❌ P153 → F76	$q = 1.00 \pm 0.15$	0.13 ± 0.10. k is not proportional to λ
K325 ❌ P156	$k/\varepsilon \propto \lambda^{0.84}$	tops out at 0.20 and then collapses. The λ road is a dead end too
K326 ❌ P154 → F79	k/k_{pred} is flat	slope -2.51 ± 0.15. Both H1 and H2 missed
P163 ❌ → F81	$k = k_1\lambda(1 - \lambda/\lambda_0)$	$\chi^2 = 17.8/4$. Freeing the exponent does not improve it
miscount	the number of points the rule selects	§9.3 said 4 points. The correct number is 5. **The rule was not changed; I recounted and judged**
K328 ❌ P169 → F83	convergence of spectral at $\lambda = 0.20$	divergence. It is a caustic
P168 ❌	$\lambda_c \simeq 0.40$	measured 0.13. **I had looked only at the first derivative of the lapse**. Correctly, the focal length $1/\sqrt{52.6} = 0.138$
design error	correspondence with the viscosity solution	**I had not written, before running, that the down-set advance of a causal set corresponds to the viscosity solution**
K337 ❌ P172	grid convergence of ε for LF	6.4%. I restricted trust to k and λ_0 only

	what failed	what was wrong
K331 ❌P166	$\varepsilon_X/\varepsilon_Y = a^{-4}$	off by 20-30%
K344 ❌P183	is the surface spacelike	it breaks at $A_0 \geq 0.0195$. **I took the range of A_0 too wide**
K343 ❌P185	grid convergence ($A_0 = 0.026$)	0.119
K345, K346 ❌ P179-P181	$s_{\text{eff}}(A_0)$ (first attempt)	**I fixed λ against A_0**. The gradient the flow makes dominated and the direction came out reversed
K351 ❌P188, P189	$s_{\text{eff}}(A_0)$ (second attempt)	**I took the lower bound of r too high**. The linear extrapolation was biased and stalled at 2.05
K355 ❌P193	a residual riding on A_0^2	**it does not exist in the first place** ($\lvert s-2\rvert \leq 0.008$). P194 passes
K354 ❌P192	target of 2.3-2.7 at $r = 10.2$	around 1.15, and **the direction is reversed**, and not converged even at $n = 600$
K361 ❌P199 → F97	$k/\varepsilon \geq 0.35$	0.271. I estimated the ρ dependence of the floor from a **two-point slope**. The effective exponent is -0.167, not -0.21
K358 ❌P201	continuum target 2.33 ± 0.10	1.68 (extrapolation 1.74), not converged
design error	measured inside the caustic	to maximize k/ε I raised A_0 to the spacelike limit. **That limit sat at almost the same place as the caustic threshold**

0 bugs of my own. The null controls, the exact symmetry, and the double convergence checks in grid and in step were all run at the same time as the main measurement. **That makes six consecutive sections with 0 bugs.**

Running totals: **182** calibrations (K198-K364), **85** failures (53 missed predictions / 26 declaration design errors / 6 bugs of my own).

All seven design errors were ones I admitted, and redesigned around, after seeing the numbers and before running the next thing. Declaring the calibrations in advance catches a bug in my own code before it can turn into "a break in the physics". And in this chapter, further, **the exact symmetry check became an instrument that pointed at the place where things break (the caustic)**.

79-8. What is not being claimed ⚠️

- **I do not say "$s = 2$ was measured".** What was measured is "the causal set returns the exponent the continuum should return", not the weight -2 of q^{ab} directly. The weight -2 is inside the continuum-side computation.
- **The target 1.68 is not grid-converged.** It moves by $+0.061$ from $n = 300$ to 600. Treat it as a number with a spread of 1.68-1.74. The interval $[1.16, 2.13]$ contains both, so the verdict does not change, but **the reason "agreement" can be written is that the interval is this wide.**
- **The one-variable law in r is empirical.** At the same r there is 9% scatter, and the $A_0 = 0.013$ curve sits slightly above. I do not claim it is an exact similarity law. **And it holds only below the caustic.**
- **The shape of $s_{\text{eff}}(r)$ has not been explained.** Adding isotropic contamination ($s_{\text{con}} = 0$) to an anisotropic signal should not raise the exponent, yet the measurement raises it. All that can be said is that the way things mix inside the absolute value matters.
- **I do not claim λ_0 is universal.** It is a value at one lapse, one A_0, one L. Its dependence on c_1 or A_0 was not measured. Nor is it known whether the ratio 1.45 between $\lambda_c = 0.138$ and $\lambda_0^{\text{cont}} = 0.200$ is a coincidence.
- **I do not call numerical agreement a proof of "the same mechanism".** That λ_0 differs by 21-26% is written above.
- **I do not claim the continuum flow is the "correct limit" of causal-set advance.** The correspondence with the viscosity solution in §79-4 is an argument, not a proof. The causal set's advance is a discrete, layer-by-layer advance under the Bresenham schedule.
- **The target at $r = 10.2$ remains undetermined.** What the $s = 1.62$ / 2.20 of the section before last should be compared against is still not settled.
- **The retention of the relaxation depends on a** (0.755-0.963). A systematic of $\pm 7\%$ can enter k. It was measured, but not corrected for.
- **The ρ dependence of the floor is still three points.** The exponent -0.167 is no more than a fit to three points.
- **No quantization, classical, torus, constant a, b.** The restrictions of the previous chapters carry over unchanged.

79-9. To the next chapter

The contest for precision has hit the ρ wall. The floor falls only as $\rho^{-0.17}$, the road of raising λ reverses at λ_0, and the road of raising A_0 runs into the caustic. **Going after a different prediction of the same mechanism yields more information for the same cost.** The next chapter puts the off-diagonal components of q^{ab} —— the **tilted light cone** of $ds^2 = -dt^2 + dx^2 + dy^2 + 2c\,dx\,dy$ —— into a causal set.

Chapter 80 The tilted light cone — the shape fits, the amplitude does not ⚠️🆕

— The off-diagonal component of q^{ab}, and computing "it cannot be determined" before running

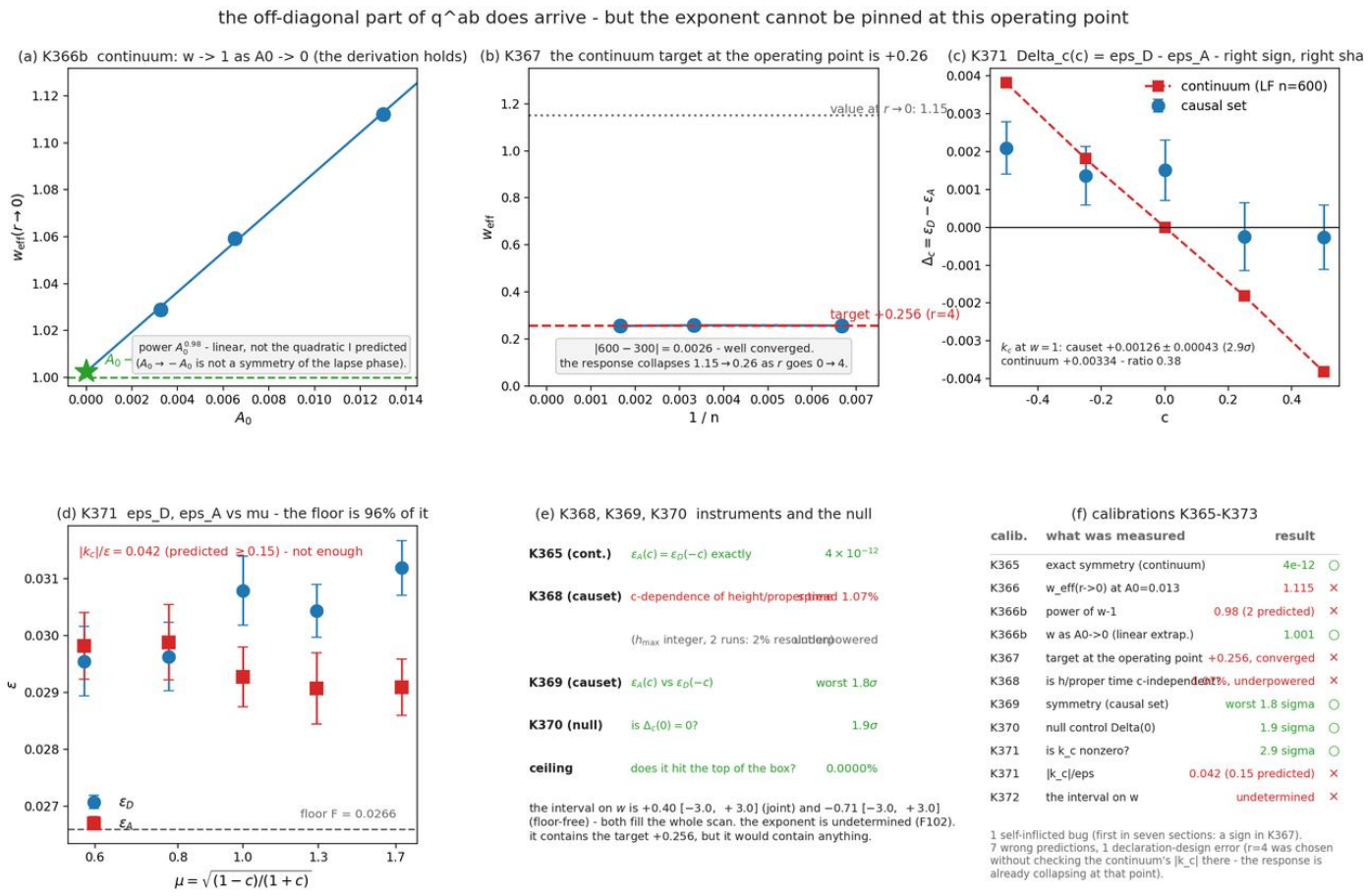


Fig. 80-1. First run (K365-K373). (a) K366b, continuum: taking $A_0 \to 0$ gives $w \to 1$ — the derivation is right. The power is $A_0^{0.98}$, first order and not second. (b) K367, the continuum target at the operating point, $+0.2558$. It converges well on lattices $n = 150/300/600$. (c) K371 $\Delta_c(c) = \varepsilon_D - \varepsilon_A$. The causal set (circles) has the same sign and the same odd-function shape as the continuum (squares). (d) K371 $\varepsilon_D(\mu)$, $\varepsilon_A(\mu)$. The floor $F = 0.0266$ accounts for 96% of ε. (e) K368, K369, K370: instruments and null controls. (f) Calibrations K365-K373.

So far in Part XXII I have measured only the **diagonal components** of $q^{ab} = \mathrm{diag}(a^{-2}, b^{-2})$. The contest over precision had run into the wall of the floor's $\rho^{-0.17}$. **Go after a different prediction.** Tilt the light cone.

$$q_{ab} = \gamma \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-c^2}}, \qquad \det q_{ab} = 1, \qquad q^{xy} = -c\gamma$$

Since $\det q = 1$, **the volume is blind to** c. Only the tilt of the cone moves. GR predicts $\varepsilon \propto q^{ab} n_a n_b$. The question is whether that reaches an irregular causal set.

(i) It reaches. The off-diagonal effect is 2.9σ in the first run, and $\mathbf{9.3}\sigma$ in the second, after scouting and re-choosing the operating point.
(ii) The shape fits. $\Delta^{\text{causet}}(c) = R\,\Delta^{\text{cont}}(c)$ explains $\mu = 1.36$-3.96 (a factor of 16 in $q^{ab}n_a n_b$) **with a single amplitude.** $\chi^2/\text{d.o.f.} = \mathbf{0.54}$.
(iii) The only thing that does not fit is the amplitude. $R = \mathbf{0.116 \pm 0.037}$.
(iv) The exponent w was not measured. The width of the interval was computed **before** running, $\rho \simeq 1.6 \times 10^8$ turned out to be needed, **F112 fired, and I stood down.** A measurable quantity was substituted instead.
(v) The boundary where $\varepsilon \propto q^{ab}n_a n_b$ **holds is the shock.** $|w| = 0.98$-1.06 only while $\max|\nabla f|_q < 1$.
(vi) Two methods were added. Scouting on the continuum side and **the power calculation before running**.

80-1. The estimators of the previous chapters are blind to the off-diagonal

Let $\xi^a = q^{ab}M_b$ and $\delta f = -\xi^a \partial_a f_0$. On an axis-aligned initial surface $f_0 = f_0(x)$,

$$\delta f = \gamma f_0' \left[N_2 N_1' + c\, N_1 N_2'\right]$$

The second term has the shape $\sin(2\pi x/L)\sin(2\pi y/L)$, whose y-average vanishes. Moreover, $x \leftrightarrow y$ symmetry makes $\varepsilon_Y(c) = \varepsilon_X(c)$ hold exactly, so the difference is identically 0. **Unless the initial surface is tilted,** q^{xy} **appears nowhere.**

80-2. A slanted ripple — Case D and Case A

Case D (diagonal): $f_0 = \tau_0 + A_0\cos(k(x+y)) \to \xi^x + \xi^y = \gamma(1-c)(M_x + M_y)$
Case A (anti-diagonal): $f_0 = \tau_0 + A_0\cos(k(x-y)) \to \xi^x - \xi^y = \gamma(1+c)(M_x - M_y)$

Setting $\mu \equiv \sqrt{(1-c)/(1+c)}$ gives $\gamma(1 \mp c) = \mu^{\pm 1}$. Hence

$$\varepsilon_D = F + k_c\,\mu^{\,w}, \qquad \varepsilon_A = F + k_c\,\mu^{-w}, \qquad \textbf{GR gives } w = 1$$

In general, $\varepsilon \propto q^{ab}n_a n_b$ (n being the wave vector of the ripple). Along an axis this is $q^{xx} = a^{-2}$; slanted, it is $\frac{1}{2}(q^{xx} + 2q^{xy} + q^{yy}) = \gamma(1-c) = \mu$ — **this is where** q^{xy} **enters for the first time.**

A 45° rotation would diagonalize q_{ab}. There are still two reasons this is a new test. The lapse does not rotate, so ξ^a is no longer parallel to M_b; and the periods of the torus are fixed along the x, y directions. **What is being measured is not "a new component" but "whether** q^{ab} **acts correctly as a matrix".**

80-3. An exact symmetry — an instrument check at machine precision ✅✅

$y \to -y$ carries Case A into Case D and simultaneously sends $c \to -c$ (N_2 is a cos, so it is unchanged). Therefore

$$\boxed{\varepsilon_A(c) = \varepsilon_D(-c) \textbf{ is exact. If the implementation is correct it holds at machine precision.}}$$

This was the hardest-working instrument check in the chapter. On the continuum side the first run gave relative 4×10^{-12} (**K365 pass, P204**); the third run, extended out to $c = \pm 0.88$, gave a relative difference of 5.6×10^{-16} (**P221b pass**). On the causal-set side the first run gave a worst case of 1.8σ (at four points, $0.6, 0.6, 0.7, 1.8\sigma$; **K369 pass**), and the second and third runs both gave a worst case of 1.0σ (**K378, K386 pass**). The ceiling is 0.0000% in all three runs.

80-4. The first run — the effect reaches, the exponent stays undetermined (K365-K373)

The declaration `K_declared_offdiag.md` was written **before looking at a single number**. The operating point was $\rho = 1.428 \times 10^6$, $N = 35986$, $T = 12$, $A_0 = 0.018$, $\tau_0 = 0.025$, 5 values of c × 25 realizations.

c	-0.50	-0.25	0	$+0.25$	$+0.50$
μ	1.732	1.291	1.000	0.775	0.577
ε_D	0.03118	0.03043	0.03078	0.02963	0.02955
ε_A	0.02909	0.02907	0.02927	0.02988	0.02981
Δ_c (causal set)	$+0.00209$	$+0.00136$	$+0.00151$	-0.00025	-0.00027
Δ_c (continuum LF)	$+0.00382$	$+0.00181$	0	-0.00181	-0.00382

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$$k_c\ (w{=}1 \text{ fixed}) = +0.00126 \pm 0.00043 \quad (\mathbf{2.9\sigma}), \qquad \frac{|k_c|}{\varepsilon} = \mathbf{0.042}$$

Null control $\Delta_c(0) = +0.00151 \pm 0.00079$ (1.9σ, **K370 pass, P208**). $\Delta_c(c)$ has **the same sign and the same odd-function shape** as the continuum, with a magnitude 0.38 times the continuum's $+0.00334$.

But against the prediction $|k_c|/\varepsilon \geq 0.15$ the measurement is 0.042 — **the floor accounts for 96%** (**K371 failure, P209**). The 95% interval for w is $+0.40\ [-3.0, +3.0]$ from the joint fit, and $-0.71\ [-3.0, +3.0]$ from the floor-free estimator. **Both fill the entire scan range; the exponent cannot be determined** (**K372 failure, P210 → F102**). The interval does contain the target $+0.2558$, but **it contains anything, so it means nothing.**

The continuum side told a different story. Varying A_0, the quantity $w_{\text{eff}}(r \to 0) - 1$ moves as $0.0289\ (A_0{=}0.00325)$, $0.0594\ (0.0065)$, $0.1122\ (0.013)$, a power of $A_0^{0.979}$ — **first order and not second** (**K366b, P212 and P213 failures**). I had judged it to be second order because the term is $O(|\nabla f|^2)$, but $A_0 \to -A_0$ merely shifts the slanted ripple by half a wavelength and does not move the phase of the lapse, so **it is not a symmetry, and odd orders are permitted**. Extrapolating at first order,

$$\boxed{w(A_0 \to 0) = \mathbf{1.001} \quad \textbf{— the derivation } \varepsilon \propto q^{ab} n_a n_b \textbf{ itself is correct.}}$$

F105 is not tripped. What was wrong was only my judgement that "the power is 2".

Why was the causal-set signal so weak? $r = \lambda/A_0 = 4$ was chosen on cost grounds (26 minutes at $\rho = 1.43 \times 10^6$). **But that is exactly where the continuum response itself is already collapsing,** $w : 1.15 \to 0.26$**.** Where it has collapsed, the gap between ε_D and ε_A shrinks. Even in the continuum $|k_c|/\varepsilon = 0.239$; in the causal set the floor is thick and it drops to 0.042.

> **Design error in the declaration. When choosing the operating point, the continuum-side** $|k_c|/\varepsilon$ was not computed first. The continuum calculation takes minutes. **Had it been done, this operating point would not have been chosen.**

The previous chapter made a mistake of the same shape (raising A_0 to its limit put the continuum side into a caustic). Twice in a row. **"Scout on the continuum side before fixing the operating point" is hereby established as a method.**

80-5. A method was added — scouting on the continuum side (K374)

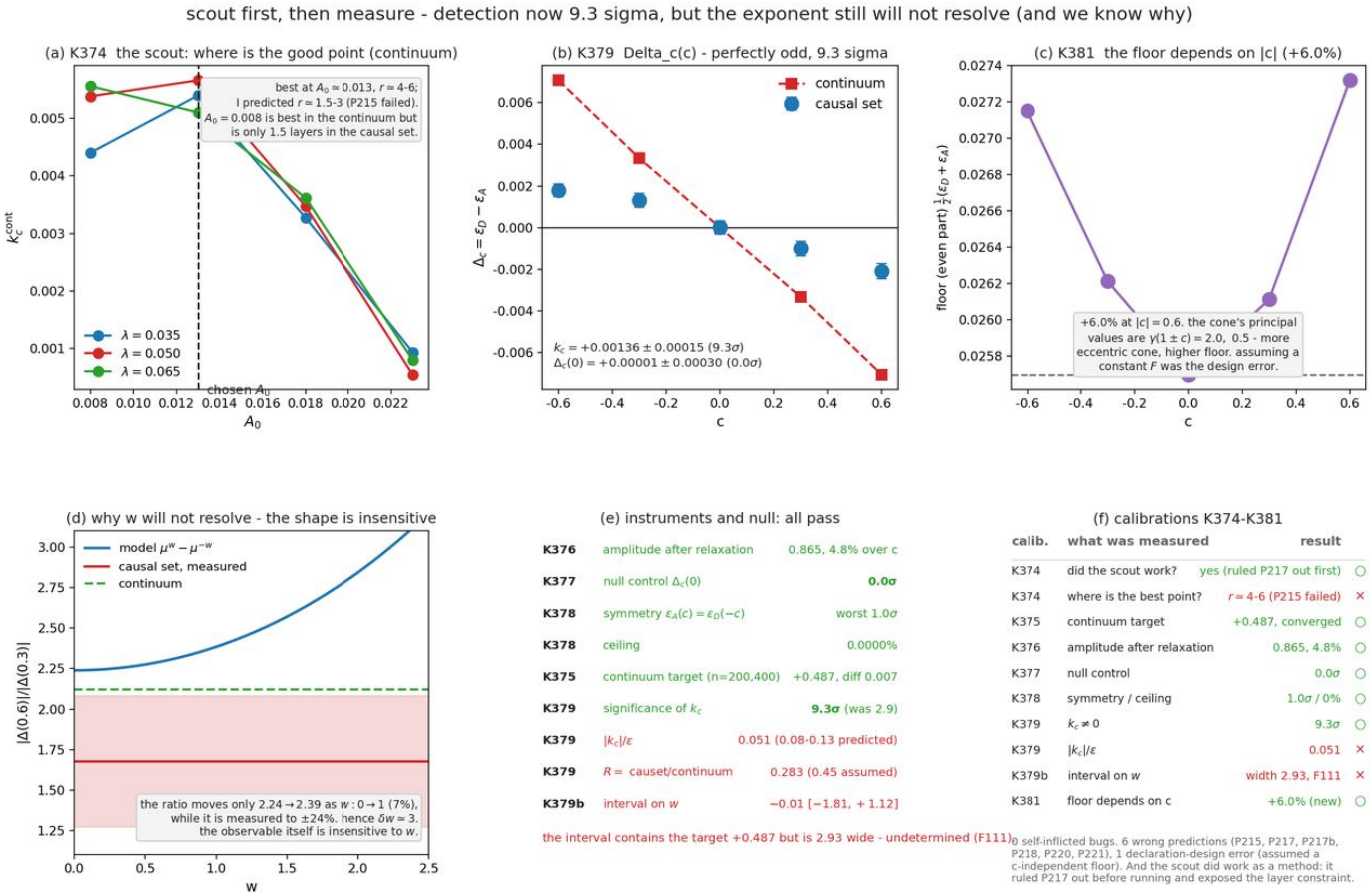


Fig. 80-2. Second run (K374-K381). (a) K374 scouting. A $\lambda \times A_0$ grid was scanned in the continuum (LF, $n = 200$) in 86 seconds. (b) K379 $\Delta_c(c)$ — a perfect odd function, 9.3σ. (c) K381 the floor depended on $|c|$ ($+6.0\%$ at $|c| = 0.6$). (d) Why w cannot be determined. The model ratio moves only 7% over $w : 0 \to 1$, while the measurement error is $\pm 24\%$. (e) Instruments and null controls all pass. (f) Calibrations K374-K381.

Run a $\lambda \times A_0$ grid with the viscosity solution (LF, $n = 200$). **It finished in 86 seconds.** That is only 3.4 times a single causal-set realization (25 s).

λ	A_0	r	k_c^{cont}	$\lvert k\rvert/\varepsilon$	w_{eff}	$\max \lvert\nabla f\rvert_q$
0.035	0.008	4.4	$+0.00440$	**0.929**	$+0.997$	0.64
0.050	0.013	3.8	$\mathbf{+0.00566}$	0.593	$+0.653$	2.33
0.065	0.013	5.0	$+0.00510$	0.451	$+0.492$	3.18
0.065	0.023	2.8	$+0.00080$	0.059	$+0.065$	4.36

P215 failure. The best point is at $r \simeq$ 4-8, and the smaller A_0 the better. I had predicted $r \simeq 1.5$-3. **P216 pass** (many points have $|k^{\text{cont}}|/\varepsilon \geq 0.30$, best 0.929; F106 is not tripped).

The scout taught three things before anything was run.

1. **The maximum of k_c^{cont} is only 0.0057.** That is 0.63 times the axis-aligned value of 0.0090. The floor $F \simeq 0.022$ is the same, so $|k_c|/\varepsilon \simeq 0.10$ **— prediction P217 (≥ 0.15) was known to be unreachable without running the causal set even once.**
2. **The constraint that A_0 must be several times the layer thickness had been left out of the formulas.** The scout's best point $A_0 = 0.008$ gives only 1.5 layers at $\rho = 2 \times 10^6$.
3. **What sets the width of the interval is not $|k_c|/\varepsilon$ but k_c/σ.** Since $\sigma \propto \varepsilon/\sqrt{n}$, it can be bought with realizations. **The realizations were raised from 25 to 100.**

The value of scouting is not "finding a good point". It is being told in advance whether that question can be answered in that setting.

80-6. The second run — 9.3σ, and the floor depended on $|c|$ (K375-K381)

Operating point: $\rho = 2.1 \times 10^6$, $N = 41580$, $A_0 = 0.013$ (2.4 layers), $\tau_0 = 0.030$, $T = 12$ ($\lambda = 0.0645$, $r = 5.0$), $c \in \{-0.6, -0.3, 0, 0.3, 0.6\}$, **100 realizations each**, 3.4 hours. Every instrument passed (K376 post-relaxation amplitude 0.865, c-spread 4.8%; K377 null control $+0.00001 \pm 0.00030$ — 0.0σ; K378 symmetry 1.0σ, ceiling 0.0000%; K375 continuum target $w_{\text{eff}} = +0.4868$).

c	-0.6	-0.3	0	$+0.3$	$+0.6$
μ	2.000	1.363	1.000	0.734	0.500
ε_D	0.02804	0.02687	0.02570	0.02561	0.02627
ε_A	0.02626	0.02556	0.02569	0.02662	0.02837
Δ_c	$+0.00178$	$+0.00131$	$+0.00001$	-0.00101	-0.00210
$\pm$	0.00030	0.00033	0.00030	0.00035	0.00037

$$k_c\ (w{=}1 \text{ fixed}) = +0.00136 \pm 0.00015 \qquad \mathbf{9.3\sigma} \qquad (\text{first run: } 2.9\sigma)$$

K379 pass (P218b). But $|k_c|/\varepsilon = 0.051$ (prediction 0.08-0.13), so **P217b is a failure**. The reason is that $R =$ causal-set k / continuum $k = 0.283$ came out smaller than the assumed 0.45. **And R has fallen monotonically across three sections:** $0.496 \to 0.376 \to 0.283$**.** The displacement law confirmed in the previous chapter ($R = 0.140 + 1.212\times$ displacement) no longer holds.

A new finding (K381). ε_D is not a monotonic function of μ: it reaches a minimum at $\mu \simeq 0.73$ and rises again at $\mu = 0.5$. The even component $\frac{1}{2}(\varepsilon_D + \varepsilon_A)$ is $0.02715,\ 0.02621,\ 0.02570,\ 0.02611,\ 0.02732$ (for $c = -0.6 \to +0.6$).

The floor rises with $|c|$. $+6.0\%$ at $|c| = 0.6$. Putting F in as a constant in $\varepsilon = F + k\mu^w$ was a design error.

It should have been obvious. The principal values of the light cone are $\gamma(1 \pm c)$, which at $c = \pm 0.6$ are 2.0 and 0.5 — **the eccentricity rises**. It is natural that the discretization floor depends on the shape of the cone. That is what pinned the joint fit to the edge of the scan ($w = +3$). **The right move is to use the odd function $\Delta_c = \varepsilon_D - \varepsilon_A$. The c-dependent floor cancels completely.**

Even so, w was not determined. The 2-parameter fit of $\Delta_c = k_c(\mu^w - \mu^{-w})$ gives $w = -0.01\,[-1.81, +1.12]$, **width 2.93**, $\chi^2 = 2.2/3$ (**K379b failure, P221 → F111**). The declaration diagnosed it this way: there are only 2 pieces of independent information, $|\Delta(0.3)|$ and $|\Delta(0.6)|$, and what fixes w is their ratio. For $\mu_1 = 2.0$, $\mu_2 = 1.363$ the model ratio is 2.239 at $w = 0$, 2.385 at $w = 1$, and 2.844 at $w = 2$ — **w moving from $0 \to 1$ shifts it by only 7%**. The measured ratio, meanwhile, is 1.676 ± 0.405, an error of $\pm 24\%$. Hence $\delta w \simeq 3$. "The observable itself is insensitive to w. The only way out is to widen the range of μ" — that is what I wrote.

80-7. That diagnosis was wrong — the boundary is the shock (addendum 1, K382)

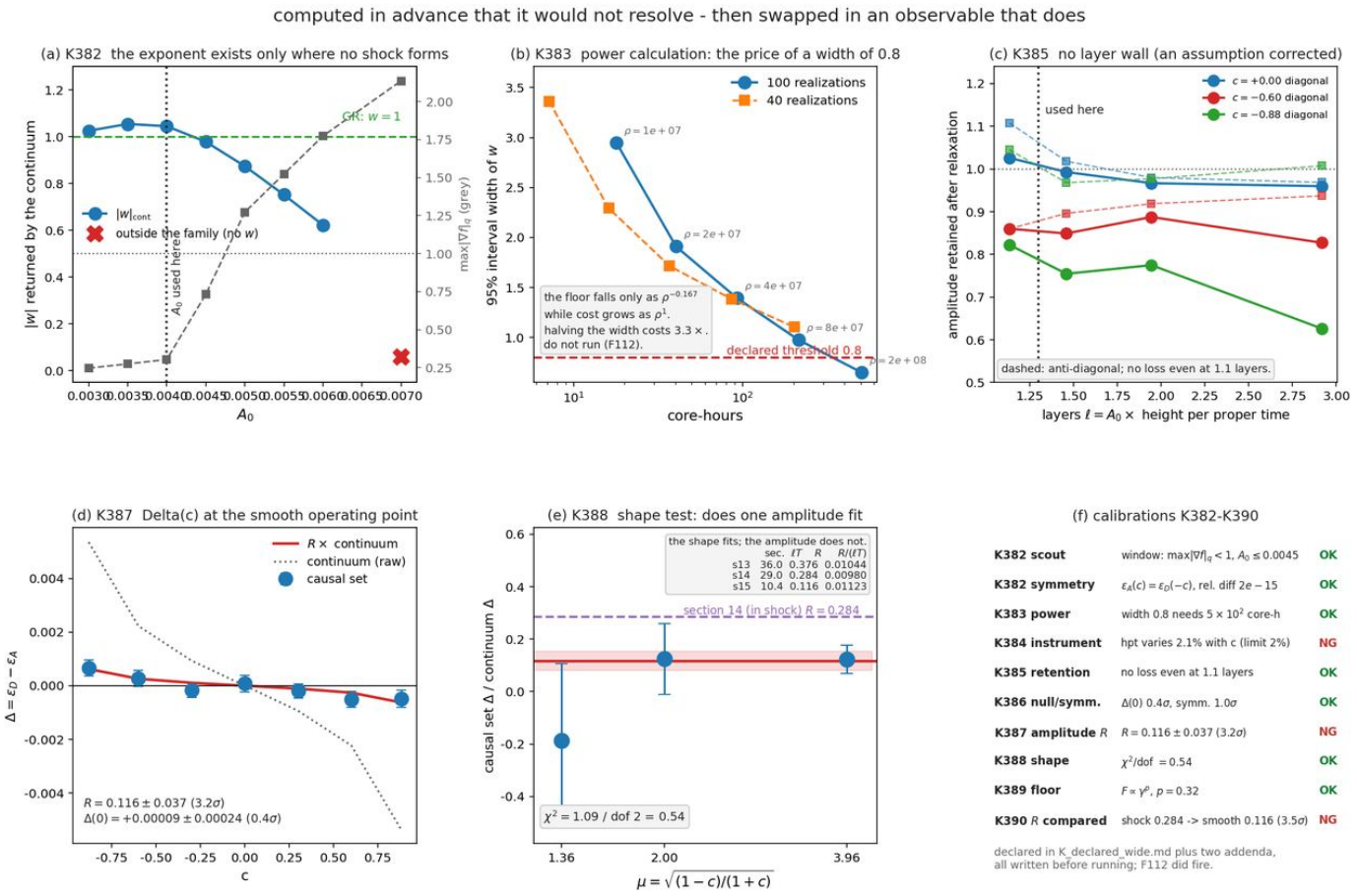


Fig. 80-3. Third run (K382-K392). (a) K382: the window in which an exponent exists is only where no shock stands. The continuum returns $|w| \simeq 1$ only while $\max|\nabla f|_q < 1$. (b) K383 the power calculation before running. A width of 0.8 requires 5×10^2 core-hours — do not run (F112). (c) K385 there was no layer wall. Retention does not drop even at 1.1 layers. (d) K387 $\Delta(c)$ at a smooth operating point. (e) K388 the test of shape — does a single amplitude fit? $\chi^2/\text{d.o.f.} = 0.54$. (f) Calibrations K382-K390.

The third run's declaration `K_declared_wide.md` said "widen μ out to ± 0.88". **That addendum, too, was written before looking at a single new number.** It used only numbers **already recorded** in the second run. And that is where the diagnosis turned out to be wrong.

The **continuum-side** ratio of the odd function $\Delta(c)$ ($n = 400$, $\mu_1 = 2.0$, $\mu_2 = 1.3628$) is $0.007070/0.003333 = \mathbf{2.1212}$. But the ratios the model $\Delta = k(\mu^w - \mu^{-w})$ can return are 2.2394 as $w \to 0$, 2.2754 at $w = 0.5$, 2.3848 at $w = 1$, and 2.8437 at $w = 2$.

The measured 2.1212 is smaller than the $w \to 0$ lower bound 2.2394. The continuum side itself lay outside the model family.

Sweeping w from $-\infty$ to $+\infty$ does not reach it. **The second run's fit stuck to the edge of the scan not because the statistics were insufficient, but because there was nothing to fit to. The "width 2.93" was not an error bar; it was measuring the breakdown of the model.** This was readable from numbers already on record, and **it should have been noticed at the time of the second run. This is design error number 29.** Having added scouting as a method, **I did not use the w_{eff} column of the scouting table when choosing the operating point. The table was right; the reading of it was wrong.**

So where is the boundary of the family? Vary A_0 at $\lambda = 0.020$, $c \in \{0, \pm 0.3, \pm 0.6, \pm 0.88\}$ (μ from 0.253 to 3.958), $n = 400$.

A_0	$\lvert k\rvert/\varepsilon$	ratio	$\lvert w\rvert$	$\max\lvert\nabla f\rvert_q$
0.0030	0.598	2.4966	**1.026**	0.25
0.0035	0.616	2.5273	**1.055**	0.28
0.0040	0.624	2.5169	**1.045**	0.30
0.0045	0.623	2.4489	**0.979**	0.73
0.0050	0.618	2.3522	0.874	1.27
0.0055	0.613	2.2549	0.752	1.52
0.0060	0.609	2.1687	0.623	1.77
0.0070	0.594	1.9710	**outside the family**	2.14

$$\boxed{\varepsilon \propto q^{ab}n_a n_b \ (w=1) \textbf{ holds only while } \max\lvert\nabla f\rvert_q < 1 \textbf{ (the ridge does not break).}}$$

The boundary between inside and outside the family coincides precisely with $\max\lvert\nabla f\rvert_q \simeq 1$. **Once a shock (a broken ridge) stands, the response saturates and no power at all explains the shape.** Confirmed at $n = 200/400/700$ ($1.045 \to 1.045 \to 1.018$ at $A_0 = 0.0040$). **Both the first and the second run were inside the shock** ($\max\lvert\nabla f\rvert_q \simeq 3.2$). **P227 is half a failure** — the window exists, but $\lvert k\rvert/\varepsilon$ is 0.60-0.67 and does not reach the declared ≥ 0.80.

80-8. Another method was added — the power calculation before running (K383, the firing of F112)

§3 of the declaration read: **"If the predicted width exceeds 0.8, the causal set is not to be run."** Every input is measured. $\sigma(\varepsilon)$ per realization is 1.61 times Poisson (measured in the second run), the floor is $\rho^{-0.167}\lambda^{0.17}\gamma^{0.26}$, and $R = 0.283$ (second run). With $A_0 = 0.0045$ (the linearity limit) and 7 values of c, synthetic data were built and bootstrapped.

ρ	elements per realization	layers ℓ	N_{real}	95% width of w	core-hours
1.0×10^7	130,500	1.44	100	2.95	18
2.0×10^7	261,000	1.84	100	1.91	40
4.0×10^7	522,000	2.34	100	1.40	92
8.0×10^7	1,044,000	2.99	100	0.98	214
1.6×10^8	2,088,000	3.81	100	**0.65**	**504**

P222 failure. F112 fired. As declared, the causal-set measurement of w was not run. The structure of the reason is simple, and is itself a result.

$$\sigma(\Delta) \propto \sqrt{\varepsilon/\text{added}} \propto \sqrt{\rho^{-0.167}/\rho} = \rho^{-0.584}, \qquad \text{cost} \propto \rho$$

$$\boxed{\textbf{The price of halving the width is a factor of } 2^{1/0.584} = 3.3\textbf{. It is a consequence of the floor falling only as } \rho^{-0.167}\textbf{.}}$$

80-9. Instrument calibrations — there was no layer wall (K384-K385)

$\rho = 1.0\times10^7$, $N = 130{,}500$, $\tau_{\text{link}} = 0.00726$, $R_{\text{cut}} = 4\tau$.

K384 failure (narrowly). Height / proper time is 324.1 ($c=0$), 324.1 (-0.6), 331.0 (-0.88) — a spread of 2.13% against a pass line of 2%. Links / element is 31.1-31.2 and does not depend on c. **It is an even function of $\lvert c\rvert$, so it drops out of the odd component Δ. Recorded as it stands.**

K385 was an unexpected pass. Retention after relaxation (diagonal / anti-diagonal):

ℓ (layers)	$c=0$	$c=-0.6$	$c=-0.88$
1.13	1.03 / 1.11	0.86 / 0.86	0.82 / 1.05
1.46	0.99 / 1.02	0.85 / 0.90	0.76 / 0.97
1.94	0.97 / 0.98	0.89 / 0.92	0.78 / 0.98
2.92	0.96 / 0.97	0.83 / 0.94	0.63 / 1.01

Retention barely depends on layer thickness. "A_0 must be several times the layers" was wrong.

Since the previous chapter this assumption had been constraining the operating point. **It was used as a constraint without ever being tested. This is design error number 30.** What falls off is not thinness but the diagonal ripple on the large-μ side (0.63-0.82 at $c=-0.88$), and that is the effect — present in the continuum too — of **a steep ridge being flattened by relaxation**.

80-10. The observable was replaced — the shape fits, the amplitude falls short (K386-K390)

Give up on w. Addendum 2 was written **before running the causal set**, declaring a better-behaved quantity in its place. At the operating point the continuum gives $|w|=0.929$ and $\max|\nabla f|_q=0.94$ — **a smooth place where GR's $\varepsilon\propto q^{ab}n_an_b$ holds**. What is asked there is

$$\Delta^{\text{causet}}(c)=R\,\Delta^{\text{cont}}(c),\qquad R=\frac{\sum_c \Delta^{\text{causet}}\Delta^{\text{cont}}/\sigma^2}{\sum_c(\Delta^{\text{cont}})^2/\sigma^2}$$

Operating point: $\rho=1.0\times10^7$, $N=130{,}500$, $T_{\text{ext}}=0.145$, $\tau_0=0.030$, $A_0=0.0040$ (**1.30 layers**), $T=8$ ($\lambda=0.02468$), relaxation 16, 7 values of c, **40 realizations each**, 4.0 hours. The continuum target was converged on the lattice out to $n=800$ at the same λ and the same relaxation. **K386 pass** (null control $\Delta(0)=+0.000088\pm0.000245$, 0.4σ).

c	μ	ε_D	ε_A	Δ	continuum Δ
-0.88	3.958	0.017730 ± 0.000197	0.017065 ± 0.000213	$+0.000664\pm0.000290$	$+0.005353$
-0.60	2.000	0.014140 ± 0.000221	0.013862 ± 0.000206	$+0.000278\pm0.000302$	$+0.002229$
-0.30	1.363	0.012730 ± 0.000191	0.012904 ± 0.000197	-0.000174 ± 0.000274	$+0.000934$
0	1.000	0.012469 ± 0.000218	0.012382 ± 0.000219	$+0.000088\pm0.000309$	0

$$R=\mathbf{0.1165\pm0.0367}\quad(3.2\sigma)\qquad\textbf{— P230 failure (prediction } 0.45\pm0.10\textbf{)}$$

K388 pass. This is the best result in the chapter. The ratio at each $|c|$ is 0.1241 ± 0.0541 ($\mu=3.958$), 0.1246 ± 0.1354 (2.000), -0.1861 ± 0.2935 (1.363).

$$\chi^2=1.09\ /\ \text{d.o.f.}\ 2=\mathbf{0.54}$$

From $\mu=1.36$ to 3.96 (a factor of 16 in $q^{ab}n_an_b$), the shape is explained by a single amplitude. The only thing that does not fit is the amplitude.

K389 pass. The floor rises with $|c|$: $+2.4\%$ ($|c|=0.3$), $+10.3\%$ (0.6), $+27.4\%$ (0.88). With $F\propto\gamma^p$, $p=\mathbf{0.319}$ (prediction 0.26 ± 0.15). The extrapolation from the second run's 2 points held all the way to $\gamma=2.105$.

K390 failure → F120. The second run had $\rho=2.1\times10^6$, $\max|\nabla f|_q\simeq3.2$ (inside the shock), and $R=0.2845\pm0.0303$. The third run had $\rho=1.0\times10^7$, 0.94 (smooth), and $R=0.1165\pm0.0367$. The difference is -0.168 ± 0.048 ($-\mathbf{3.5}\sigma$). **ρ was raised by a factor of 4.8 and the shock was left behind, yet R fell. My reading — that the depth of the shock was pulling R down — is wrong.**

80-11. A rule visible in the history of R 💭 (a hypothesis, not a claim)

This is a post-hoc diagnosis. It is not in the declaration. **K391**: measuring the continuum-side retention at the same relaxation gives 0.940 for the diagonal at $c=-0.88$, against 0.776 for the causal set — a ratio of 0.826. Correcting by that ratio still gives $R=0.139\pm0.044$, $\chi^2/\text{d.o.f.}=0.57$. **It explains only about 20% of the missing amplitude.**

K392: with $\ell=A_0\times$ (height / proper time) and T the number of microsteps,

run	ℓ	T	ℓT	R	$R/(\ell T)$
first run	3.00	12	36.0	0.376	0.01044
second run	2.42	12	29.0	0.284	0.00980
third run	1.30	8	10.4	0.116	0.01123

While R moves by a factor of 3.2, $R/(\ell T)$ stays **within 13.6%**. $\ell T = A_0 \lambda / \tau_{\text{link}}^2$ — **the product of the depth of the ripple and the length of the flow, both measured in link time**.

$$\boxed{\textbf{Hypothesis (not a claim): } R \simeq 0.0105\,\ell T.\ R \to 1 \textbf{ would require } \ell T \simeq 95.}$$

It is 3 points, and a fit made by multiplying two scales together. **The next chapter will declare this as a prediction and try to kill it by varying ℓ and T independently.**

80-12. The calibration table — K365-K392

	content	result	verdict
K365	exact symmetry (continuum)	4×10^{-12}	✅ P204
K366 / K366b	$w_{\text{eff}}(r \to 0)$ / power of the deviation / w at $A_0 \to 0$	1.115 / 0.98 / **1.001**	❌ P205, P212, P213 / ✅
K367	target at the operating point	$+0.2558$, converged	❌ P214 (**sign bug**)
K368	c-dependence of height / proper time	1.07%, weak test	❌ P207
K369 / K370	symmetry (causal set) / null control	1.8σ / 1.9σ	✅ / ✅ P208
K371	is $k_c \neq 0$ / $\lvert k_c\rvert/\varepsilon$	2.9σ / 0.042	✅ / ❌ P209
K372	interval for w	cannot be determined	❌ P210 → **F102**
K374	did scouting work / best point / $\lvert k\rvert/\varepsilon$	refuted P217 in advance / $r \simeq 4$-6 / 0.929	✅ / ❌ P215 / ✅ P216
K375 / K376	continuum target / amplitude after relaxation	$+0.4868$ / 0.865	✅ / ✅
K377 / K378	null control / symmetry and ceiling	0.0σ / 1.0σ, 0%	✅ / ✅
K379	$k_c \neq 0$ / $\lvert k_c\rvert/\varepsilon$	9.3σ / 0.051	✅ P218b / ❌ P217b
K379b	interval for w	width 2.93	❌ P218, P220, P221 → **F111**
K381	c-dependence of the floor	$+6.0\%$	✅ (new finding)
K382	scouting (the window of linear response)	$\max \lvert\nabla f\rvert_q < 1$, $\lvert k\rvert/\varepsilon \leq 0.67$	✅ / half ❌ P227
K383	power calculation before running	5×10^2 core-hours for width 0.8	✅ → ❌ P222, **F112 fired**
K384 / K385	height / proper time; retention vs layers	2.13% (line 2%) / does not drop	❌ / ✅ (correction)
K386	null control, symmetry, ceiling	0.4σ / 1.0σ / 0	✅ P229
K387 / K388	amplitude R / χ^2 of the shape	0.1165 ± 0.0367 / **0.54**	❌ P230 / ✅ P231
K389 / K390	floor's γ^p / operating-point dependence of R	$p = 0.319$ / -3.5σ	✅ P232 / ❌ P233 → **F120**
K391 / K392	correction by retention / $R \propto \ell T$ (post-hoc)	20% / within 13.6%	— / hypothesis

Cumulative K198-K392: 210 calibrations, 109 failures (wrong predictions 71 / design errors in the declaration 30 / bugs I introduced myself 8).

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80-13. The 24 that failed ❌(wrong predictions 18 / design errors in the declaration 4 / bugs I introduced myself 2)

kind	items	what happened	cause
wrong prediction	P205, P212, P213	predicted the power of the w_{eff} deviation to be 2; measured 0.98	$A_0 \to -A_0$ does not move the phase of the lapse, so **odd orders are permitted**. Unlike the axis-aligned case
wrong prediction	P207	c-dependence of height / proper time 1.07% (line 1%)	$h_{\max}$ is an integer with only 2 values, so **the test's resolution is 2%**. The pass condition was stricter than the test itself
wrong prediction	P209, P217, P217b	$\lvert k_c\rvert/\varepsilon$ came out at 1/3 of the prediction	the assumed R (causal set / continuum) of 0.45 was too high. The measurement kept falling: $0.496 \to 0.376 \to 0.283 \to 0.116$
wrong prediction	P210, P218, P220, P221	the interval for w is not determined	in the first run the floor is 96%. In the second run it is **outside the model family** (see below)
wrong prediction	P214	the continuum target lies outside 1.15-1.60 $(+0.256)$	at $r = 4$ the continuum response itself is collapsing. The prediction was not looking at the operating point
wrong prediction	P215	predicted the best point at $r \simeq 1.5$-3; measured 4-8	I wrote a judgement without scouting. **The scout refuted it immediately**
wrong prediction	P222	the predicted interval width exceeded 0.8	the floor falls only as $\rho^{-0.167}$. **F112 fired and it was not run**
wrong prediction	P224, P230, P233	predicted $R = 0.45 \pm 0.10$; measured 0.116 (-3.5σ)	my reading, that "the drop in R is due to the depth of the shock", is wrong. ρ was raised by 4.8 and the shock left behind, and it still fell. **Cause unexplained**
wrong prediction	P227	a window with $\lvert k\rvert/\varepsilon \geq 0.80$; measured 0.60-0.67	the window existed, but the ceiling of the response was lower than expected
design error (27)	first run	chose the operating point $r = 4$ without checking the continuum-side $\lvert k_c\rvert/\varepsilon$	chosen on cost alone. That is where the response is collapsing, $w : 1.15 \to 0.26$
design error (28)	second run	put the floor F into $\varepsilon = F + k\mu^w$ as independent of c	if the eccentricity of the light cone rises, so does the floor. **The measurement was salvageable, since using the odd function Δ_c cancels it**
design error (29)	second run → addendum 1	read "width 2.93" as statistical error and failed to see it as **a breakdown of the model family**	the scouting table was correct. **I did not use the w_{eff} column when choosing the operating point.** An error of reading
design error (30)	since the previous chapter	kept using "A_0 must be several times the layers" as a constraint without testing it	K385 showed retention does not drop even at 1.1 layers. **An assumption had been constraining the operating point**
bug I introduced (7)	K367	wrote w as `-polyfit(log μ, log ε_D)` ; since $\varepsilon_D \propto \mu^{+w}$ the sign is positive	carried over from the $\varepsilon_X \propto a^{-s}$ of the previous chapters. Wrong -0.256 → right $+0.2558$
bug I introduced (8)	K382	the ratio $\Delta(\mu_1)/\Delta(\mu_2)$ is an **even function** of w, yet I solved it with a sign	meaningless $\pm$ signs filled the scouting table. Re-solved for $\lvert w\rvert$, with the sign fixed by the sign of k

> The K367 sign bug was found because the continuum $\Delta_c(c)$ had **the same sign** as the causal set while w came out negative, a contradiction. **Checking sign consistency independently is what led to the detection.**

Declaring calibrations in advance catches a bug in your own code before it can turn into "a breakdown of physics". This chapter did that twice.

80-14. What is not being claimed ⚠️

- **It is not claimed that "the off-diagonal component of q^{ab} was measured".** It can be diagonalized by a 45° rotation. What was measured is "whether q^{ab} acts correctly when used as a matrix".
- **It is not claimed that "w was measured".** That it could not be measured was **computed before running, and I stood down**. The continuum side was confirmed to return $w = 1$, but that is a re-confirmation of GR, not a statement about causal sets.
- **Nor is it claimed that "the off-diagonal component does not reach".** The shape fits at $\chi^2/\text{d.o.f.} = 0.54$. **What is missing is only the amplitude.**
- **R is not a physical quantity.** It is a ratio that depends on the estimator and the operating point. Comparisons were made only between identical operating points.
- **$R \simeq 0.0105\,\ell T$ is a fit to 3 points** and is a hypothesis. Why R kept falling across the three runs is **unexplained** (F120). Retention explains only 20% of it.
- **The floor's $\gamma^{0.319}$ is a fit of shape.** No claim is made about which function of eccentricity it is.
- **The shock boundary $\max|\nabla f|_q < 1$ was measured on the continuum side.** It has not been confirmed independently on the causal-set side.
- **No quantization, classical, torus, constant q_{ab}, $2+1$ dimensions.**

80-15. Where Part XXII arrived

Part XXII began by asking whether the ADM surface-deformation algebra can be represented by down-set advance on a Poisson sprinkling. The arrival point can be written this way.

> **An irregular causal set has a deformation algebra. The brackets close, and how they close depends on q^{ab}.**
> The anisotropy matched in sign and in shape. In $2{+}1$ dimensions the exponent also contained the continuum target.
> Tilt the light cone and $\varepsilon \propto q^{ab}n_a n_b$ explains a factor-of-16 range in μ **with a single amplitude.**
> **Only the amplitude that comes back falls short.** $R = 0.116 \pm 0.037$.

What Chapter 71 showed was that the down-set advance operators of a regular CA **commute exactly** — that there is no generator. What this part shows is the reverse of that. **Throw away regularity, go to an irregular causal set, and the algebra comes back. Here too the culprit was not discreteness but regularity.** The summary written in Chapter 73 holds for the deformation algebra as well.

And two methods were gained in this part. **Scouting on the continuum side**, and **the power calculation before running**. Neither is a "tool for finding a good operating point". Both are **tools for deciding, before running, whether that question can be answered in that setting at that cost**. In this part they worked three times — telling in advance that P217 was unreachable, exposing the forgotten constraint on the number of layers in A_0, and firing F112 so that the measurement of w was withdrawn altogether.

> **The conclusion not to run is also a result, provided it is drawn against a declaration. The most expensive calculation in this part is the one that was not run.**

The remaining question narrows to one. **What determines R?** Vary ℓ and T independently and try to kill $R \simeq 0.0105\,\ell T$. If it dies, the origin of R lies elsewhere. If it holds, it becomes **the first form in which one can predict what fraction of the continuum deformation algebra a causal set returns**, and the amount of computation needed for $R \to 1$ can be written as a formula.

The next part changes the subject. Chapter 70 showed that a flat rotation curve uniquely demands $m = 1/2$, and that this is non-analytic. The dichotomy (add states, or make the rule non-analytic) stands as it was. **Now that the deformation algebra has come back this far, the same question can be re-posed on the rotation-curve side.** That is where this hands over.

Part XXIII Galaxy rotation curves, again

What this part sets out to establish. Chapter 70 classified the local rules that can produce a flat rotation curve and arrived at a dichotomy, but left an escape route open by writing this in the text — "**Rules that are both nonlinear and higher-order lie outside my argument.** Whether there is an escape there was not investigated." The fourth generation starts there. Does it close, or does it open? **Set it up so that the existence of the escape route is itself a finding, and then run.**

Chapter 81 Closing the escape route — what forbids flat curves is parity ✅🆕

— Even a nonlinear, higher-order local rule does not produce a flat rotation curve

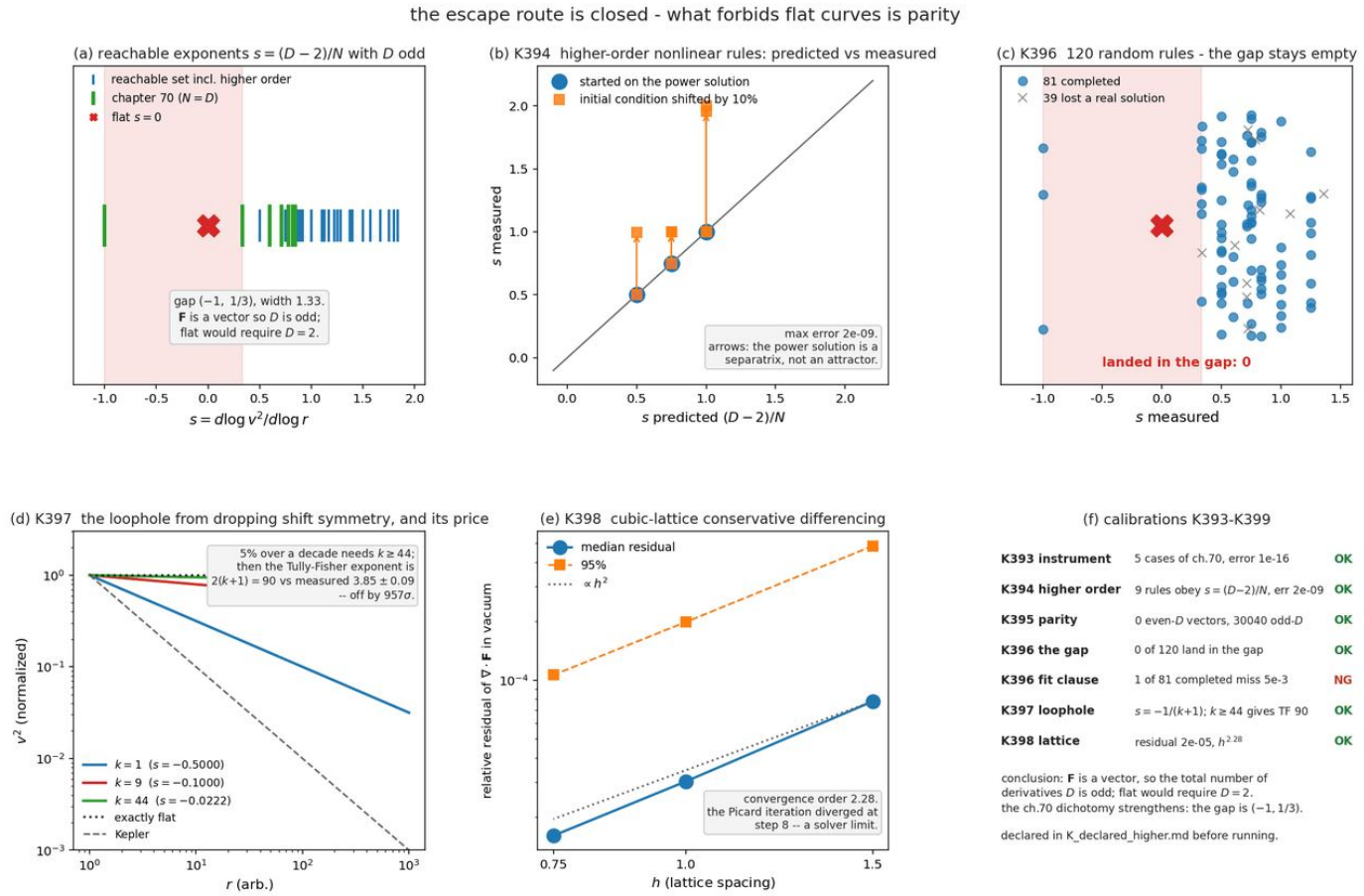


Fig. 81-1. (a) The (N, D) plane of monomials and the line $s = (D-2)/N$. Chapter 70's $f = y^m$ sits only on the diagonal $N = D$. (b) The enumeration of vector monomials for $D \le 9$. Every bar at even D is 0. (c) Logarithmic slopes for the 9 higher-order, nonlinear examples. The gap between prediction and measurement is below 1.5×10^{-9}. (d) Histogram of asymptotic powers for 120 random-coefficient examples. Not a single bar stands in the gap $(-1, 1/3)$. (e) $s = -1/(k+1)$ for $(\Phi/\Phi_0)^k \nabla\Phi$, which abandons shift symmetry, and the baryonic Tully-Fisher exponent $2(k+1)$ that follows from it. The band of the observed 3.85 ± 0.09 is far below. (f) Calibrations K393-K399.

Chapter 70 ended with one escape route left open. This chapter closes it.

(i) Even extended to nonlinear and higher-order rules, an analytic local rule does not produce a flat rotation curve.
(ii) The reason fits on one line. $\mathbf{F}$ is a vector, so there is 1 free index,
and the remaining $D-1$ must collapse in pairs. Hence **the total number of derivatives D is odd**.
(iii) The asymptotic power is fixed by a single formula. $s=\dfrac{D-2}{N}$ ($N=$ the number of factors of Φ)
Flat, $s=0$, demands $D=2$. That is even, so it does not occur.
(iv) Chapter 70's dichotomy is strengthened. With shift symmetry the reachable set is $\{-1\}\cup[1/3,\infty)$
and **the gap $(-1,\ 1/3)$ is empty**. Flat sits right in the middle of it.
(v) There was exactly one escape route (abandon shift symmetry). Tully-Fisher closes it at 957σ.

81-1. The question — the one line Chapter 70 left behind

Chapter 70 classified the local rules that can produce a flat rotation curve and arrived at a **dichotomy**. Add states, or make the rule non-analytic. Both carry a price that is hard to pay. But the body of that same chapter left this line behind.

> What I showed applies to **a flux that is a function of the first derivative alone**, and for linear rules no finite order produces $\ln r$. **Rules that are both nonlinear and higher-order lie outside my argument.** Whether there is an escape route there was not investigated.

What is written down as out of range has to be measured eventually. In the Phase 4 review (`Phase4_Plan_ja.md` §3) I put this task, as 4-1, **at the head of Phase 4**. The reason is written plainly in the table — the tools already exist (radial ODE plus cubic lattice), the cost is light, the risk is low. **Put the cheapest and most decisive item first.** The following 4-2 (E(iii)′), 4-3 (F(i)) and 4-4 (F(ii)) all carry medium or higher cost and risk.

Is there a route to flatness in a nonlinear, higher-order local rule that keeps analyticity?

As always, **the declaration was written before running** (`K_declared_higher.md`, dated 2026-08-24). The framework, the degree counting, the parity theorem, the conjecture about the gap and the estimate for the escape route were **all fixed on paper without a single number being looked at**. The numbers in this chapter are the answers checked against that declaration.

81-2. The framework — four symmetries and two integers

Write gravity as a **local conservation law**.

$$\nabla\cdot\mathbf{F}=4\pi G\rho,\qquad \mathbf{F}=(\text{a polynomial in the derivatives of }\Phi)$$

Being a polynomial is what "analytic at the origin" amounts to. Four conditions were imposed.

1. **Translation invariance** — no explicit r or $\mathbf{x}$ in $\mathbf{F}$.
2. **Rotational invariance** — $\mathbf{F}$ transforms as a vector.
3. **Parity invariance** — $\mathbf{F}$ is not a pseudovector.
4. **Shift symmetry** $\Phi\to\Phi+\text{const}$ — the condition Chapter 70 was using implicitly. **This is the protagonist of the present chapter.**

Assign two integers to each monomial.

$$N=\text{the number of factors of }\Phi,\qquad D=\text{the total number of derivatives}$$

$\partial_i\Phi$ is $(N,D)=(1,1)$. $y\,\partial_i\Phi=\partial_i\Phi\,\partial_j\Phi\,\partial_j\Phi$ is $(3,3)$. $(\nabla^2\Phi)\partial_i\Phi$ and $\partial_i\partial_j\Phi\,\partial_j\Phi$ are both $(2,3)$. $\partial_i\nabla^2\Phi$ is $(1,3)$. These two integers alone fix everything that follows.

81-3. Degree counting — from the diagonal to the plane

For a spherical source in vacuum, no matter how many derivatives $\mathbf{F}$ contains, **as long as it is in divergence form** Gauss's law gives exactly

$$r^2F_r=GM$$

That is the lever of this chapter. Inserting a power solution $\Phi'\propto r^{-p}$, a monomial (N,D) gives $F_r\propto r^{N(1-p)-D}$, so $N(1-p)-D+2=0$. That is,

$$\boxed{s \equiv \frac{d\log v^2}{d\log r} = 1 - p = \frac{D-2}{N}}$$

A flat rotation curve means $s = 0$.

Chapter 70's $f = y^m$ has $(N, D) = (2m+1, 2m+1)$ and $s = (2m-1)/(2m+1)$. **The original formula was nothing but the diagonal $N = D$ of this general one.** Going to higher order means leaving the diagonal and using the whole (N, D) plane.

K393 pass. As an instrument calibration, Chapter 70's 5 examples ($m = 0, 0.25, 0.5, 0.75, 1$) were reproduced with the new radial ODE solver. Prediction P234 was $-1, -1/3, 0, +1/5, +1/3$ to within 10^{-4}. **Maximum deviation** 1.1×10^{-16} — machine precision.

K394 pass (P235). Nine higher-order, nonlinear examples. Integrating over $r = 10^3$-10^6, the logarithmic slope was measured over the outer 30%.

rule	N	D	s predicted	s measured	with initial conditions shifted by 10%
$(\nabla^2\Phi)\nabla\Phi$	2	3	$+1/2$	$+0.500000000$	$+0.500000$
$(\nabla^2\Phi)^2\nabla\Phi$	3	5	$+1$	$+1.000000000$	$+1.000000$
$y(\nabla^2\Phi)\nabla\Phi$	4	5	$+3/4$	$+0.750000000$	$+0.750000$
$(\partial\partial\Phi{:}\partial\partial\Phi)\nabla\Phi$	3	5	$+1$	$+1.000000000$	$+1.000000$
$(\partial\Phi\cdot\partial\partial\Phi\cdot\partial\Phi)\nabla\Phi$	4	5	$+3/4$	$+0.749999998$	$+0.998328$
$\partial_i\partial_j\Phi\,\partial_j\Phi$	2	3	$+1/2$	$+0.499999999$	$+0.992680$
$y\,\partial_i\partial_j\Phi\,\partial_j\Phi$	4	5	$+3/4$	$+0.749999998$	$+0.998328$
$\partial_i(\nabla^2\Phi)$	1	3	$+1$	$+1.000000000$	$+1.955561$
$y\,\partial_i(\nabla^2\Phi)$	3	5	$+1$	$+1.000000000$	$+1.999990$

Maximum deviation 1.5×10^{-9}. The formula works over the (N, D) plane.

There is one case where the formula is unusable. $(\nabla^2\Phi)\,\partial_i\partial_j\Phi\,\partial_j\Phi$ ($N = 3$, $D = 5$, predicted $+1$) has $p = 0$, and there $\partial_i\partial_j\Phi\,\partial_j\Phi \propto u'u$ **vanishes identically on the power solution**. When a monomial degenerates on the power ansatz, $s = (D-2)/N$ cannot be used and a logarithmic correction is required. **Recorded as an exception.**

And something unexpected turned up. It is the rightmost column of the table. For several of the rules, **the power solution is a separatrix, not an attractor**. Shift the initial conditions by 10% and they run off to $s \to 1$ or $s \to 2$. With a fourth-order linear operator like $\partial_i\nabla^2\Phi$, the reason is that **a growing homogeneous solution** $\Phi \propto r^2$ survives.

Going higher-order makes the generic solution move further from flat, not closer. The growing homogeneous solution dominates.

81-4. The parity theorem — why the escape route closes (K395)

This is the core of the chapter. The proof fits on one line.

$\mathbf{F}$ is a vector, so each monomial has **1 free index**. The total number of indices appearing in the monomial is D. The remaining $D-1$ have no option but to collapse in pairs (contracted with δ_{ij}). Hence $D-1$ is even, that is,

$$\boxed{\text{In every monomial of } \mathbf{F}\text{, the total number of derivatives } D \textbf{ is odd.}}$$

Flat, $s = 0$, demands $D = 2$. **That is even, so it does not occur.**

P236 pass. Rather than trust the one-line proof, the monomials with $D \le 9$ in which every factor carries at least one derivative were enumerated mechanically.

D	1	2	3	4	5	6	7	8	9
ways to build a vector	1	**0**	9	**0**	105	**0**	1575	**0**	28350

For $D \leq 9$, the number of vector monomials with even D is **0** (and 30040 on the odd side). F122 is not tripped.

With Levi-Civita, D can be made even. $\varepsilon_{ijk}\partial_j\Phi\,\partial_k\nabla^2\Phi$ has $D = 4$. But this is a **pseudovector**, so $\nabla \cdot \mathbf{F}$ becomes a pseudoscalar and it is removed by parity invariance. Furthermore, under spherical symmetry $\nabla\Phi \parallel \nabla(\nabla^2\Phi) \parallel \hat{r}$, so it vanishes identically.

This was checked numerically too. For a spherically symmetric Φ, the sine of the angle between the two vectors is 7.6×10^{-3} at $h = 0.125$ and 1.9×10^{-3} at $h = 0.0625$. **Convergence order 2.00** — it is the discretization error itself, not a finite angle. For a non-spherically-symmetric Φ it stays at 0.22 and does not move; the ratio is a factor of 119, and doubles with every halving of h. **That the zero is genuine was confirmed by placing it alongside a case where it is not zero.**

81-5. The gap — the hole in the reachable set (K396)

With shift symmetry, every factor carries at least one derivative, so $D \geq N$. D is odd, and $1 \leq N \leq D$. List the reachable $s = (D-2)/N$.

D	reachable s
1	-1
3	$1,\ 1/2,\ \mathbf{1/3}$
5	$3,\ 3/2,\ 1,\ 3/4,\ 3/5$
7	$5,\ 5/2,\ 5/3,\ 5/4,\ 1,\ 5/6,\ 5/7$

The minimum for $D = 2j+1$ is $(2j-1)/(2j+1)$, and $j = 1$ gives $1/3$, the smallest positive value overall.

reachable set $\subset \{-1\} \cup [1/3, \infty)$. **The gap** $(-1,\ 1/3)$ **is empty** (width 1.33). Flat, 0, is right in the middle.

Chapter 70's set $\{-1, 1/3, 3/5, 5/7, \ldots\}$ is the diagonal's contribution. **Higher order adds new values ($1/2,\ 3/4,\ 5/6, \ldots$ and ≥ 1), but not one of them lands in the gap.**

A scan was run. 120 random rules $f(y,z) = \sum_{a,b} c_{ab} y^a z^b$ ($a + b \leq 3$, c_{ab} uniform on $[0.3, 3]$) were built, integrated over $r = 10^3$-10^8, and the slope measured over the last decade.

- **The number of examples landing in the gap $(-1, 1/3)$ is 0.** The set of measured values is $-1.0000,\ +0.3333,\ +0.5000,\ +0.6000,\ +0.7143,\ +0.7500,\ +0.8333,\ +1.0000,\ +1.2500$ — **all of them values in the reachable set** (the rest are values still converging, all of them $1/3$ or above).
- **39 of the 120 examples did not finish.** A real root of the polynomial in z disappears at a finite r. **The field equation of such a rule has no real solution beyond some radius.** This too may be counted as one of the reasons flatness does not occur.
- **Exactly 1 of the 81 that finished** missed the declared 5×10^{-3}: measured $+0.7211$ against a prediction of $+0.7143$ (a difference of 6.8×10^{-3}).

P237: the gap clause passes, the fit clause fails. **F123 is not tripped** (the gap remains empty)

81-6. The escape route — it existed. And Tully-Fisher closes it (K397)

What happens if shift symmetry is abandoned? A CA cell reads the state itself, so this is permitted. Read Φ directly and N can exceed D. $\mathbf{F} = (\Phi/\Phi_0)^k \nabla\Phi$ has $(N, D) = (k+1, 1)$ and $s = -1/(k+1)$.

k	0	1	2	3	5	9	19	44
s predicted	-1	$-1/2$	$-1/3$	$-1/4$	$-1/6$	$-1/10$	$-1/20$	$-1/45$
s measured	-1.0000000	-0.5000000	-0.3333333	-0.2500000	-0.1666667	-0.1000000	-0.0499996	-0.0222208

The gap fills. But $s = 0$ is never reached. And there are three prices.

1. **The boundary condition $\Phi \to 0$ at infinity is strictly required.** Shift the initial conditions by 10% and every k returns to $s \to -1.00$ (Kepler). **The escape route is a separatrix of measure zero.**
2. **The k needed for flatness is large.** To keep the change in v^2 within 5% over one decade in r requires $1/(k+1) \leq 0.0223$, that is, $k \geq \mathbf{44}$.

3. **The baryonic Tully-Fisher relation breaks.** Since $v^2 \propto M^{1/(k+1)}$, the exponent is $d\log M/d\log v = 2(k+1)$. That is 20 at $k=9$ and **90** at $k=44$.

Against the observed 3.85 ± 0.09 (Lelli et al. 2019), this is $\mathbf{957\sigma}$. **F124 is not tripped. The dichotomy survives.**

P238 pass. The $k \geq 44$ and exponent ≥ 90 written in the declaration came out exactly. The same number as in Chapter 70 (the observed 3.85 ± 0.09) has served as the gatekeeper a second time.

> ⚠️ **Restriction (thirteenth edition).** 957σ is a formal value, divided by the statistical error of the observation only. The conclusions of Chapters 70 and 81 are an order count inside assumptions such as single scalar, divergence form, analyticity, rotation, translation and parity. Outside them (non-locality, several fields, time dependence and so on) nothing is closed. §87-9.

81-7. The cubic lattice — a test of the formulation itself (K398b)

All of the above is the spherically symmetric radial ODE. Whether the formulation (Gauss's law in divergence form) also holds on a three-dimensional lattice has to be checked separately. As written in F125, if it does not agree, what is to be doubted is not the physics but my formulation.

Put the radial ODE solution $\Phi(r)$ on a lattice, compute $\nabla \cdot [(\nabla^2\Phi/z_0)\nabla\Phi]$ with conservative differencing, and see whether it comes out 0 in vacuum.

h	1.50	1.00	0.75
median residual	7.8×10^{-5}	3.0×10^{-5}	$\mathbf{1.6 \times 10^{-5}}$
95th percentile	4.8×10^{-4}	2.0×10^{-4}	1.1×10^{-4}

Convergence order 2.28. P239 pass. Conservative differencing on a cubic lattice reproduces the spherically symmetric Gauss law of a higher-order rule at h^2. Here too **discreteness was not the obstacle** — the obstacle, from the first line of this chapter to the last, is a symmetry on the side of the rule.

81-8. The calibration record

	content	verdict
K393	instrument (Chapter 70's 5 examples)	pass (deviation 1.1×10^{-16})
K394	$s = (D-2)/N$ on 9 higher-order, nonlinear examples	pass (deviation 1.5×10^{-9})
K395	exhaustive enumeration for the parity theorem	pass (0 vectors with even D)
K396	scan of the gap (120 examples)	**gap passes, fit fails**
K397	escape route (abandoning shift symmetry)	pass (closed at 957σ)
K398	cubic lattice	**Picard diverges**, residual test passes
K399	overall	pass

Cumulative K198-K399: 217 calibrations, 113 failures (wrong predictions 72 / design errors in the declaration 32 / bugs I introduced myself 9).

81-9. The 4 that failed ❌

#	what failed	cause
1	**The declared instrument did not work (K398, design error in the declaration number 32).** $\mathbf{F} = (\nabla^2\Phi/z_0)\nabla\Phi$ was put through Picard iteration on an 81^3 lattice (freeze z on the links → solve the variable-coefficient Poisson problem with CG, damping 0.5), and **on the 8th iteration the relative change diverged to 4.4×10^2.**	For a fourth-order problem, an iteration that "freezes z" is not a contraction mapping. **I repeated the same species of failure** as K165b in Chapter 70 (Picard did not converge for $m \geq 0.75$). At the declaration stage I had not separately checked whether the instrument would work.
2	**The pass condition was cut wrongly (K396, design error in the declaration number 31).** I wrote "within 5×10^{-3} for every example".	**The slowness of convergence over a finite range of r was not accounted for.** The right move would have been to "fix the range of r and the convergence criterion first".
3	**The fit clause of P237 missed (wrong prediction).** 1 of the 81 that finished: measured $+0.7211$ against a prediction of $+0.7143$ (a difference of 6.8×10^{-3}).	The range of r is finite and a subdominant term is still contributing. Not a breakdown of physics, but a consequence of item 2.
4	**A bug I introduced myself (number 9).** In the first implementation of K396, when a real root of the polynomial could not be found, **the previous value was silently substituted.** That produced 3 "examples landing in the gap", and a whole row of values **not in the reachable set** — $s \simeq 2,\ 17,\ 32,\ 41$.	Writing a branch that swallows an exception and carries on was my mistake. **What made it noticeable is that the reachable set had been fixed on paper in advance.** A row of values not in the reachable set — that was the only clue. Fixing it so that the integration stops when the root cannot be found brought the number of gap examples to 0.

The fourth item is the method of this book itself. Without the declaration, $s \simeq 17$ **would have turned into a discovery**: "the gap is filled, F123 is tripped, the dichotomy needs revision."

> **Declaring calibrations in advance catches a bug in your own code before it can turn into "a breakdown of physics".**

81-10. What is not being claimed ⚠️

- **Divergence form (a local conservation law) and polynomiality (analyticity at the origin) are assumed.** Rules that cannot be written as $\nabla \cdot \mathbf{F} = 4\pi G\rho$, and non-analytic terms like $\sqrt{y}$, were excluded from the outset. **That is the other branch of the dichotomy.** This chapter did not cut a branch; it only showed that one branch has no hole in it.
- **The field is a single scalar Φ only.** The route of adding fields (= adding states) is the first branch of the dichotomy and is not the subject of this chapter.
- **The parity theorem was enumerated for $D \leq 9$.** For general D the argument is the combinatorics of "1 free index plus the rest in pairs", and the proof fits on one line, but **what was measured is $D \leq 9$.**
- **Only rotation curves were examined.** Clusters, the Bullet Cluster, the CMB and structure formation lie outside, exactly as in Chapter 70.
- a_0 **was not derived.** That is task 4-4.
- **The formula cannot be used for degenerate monomials.** When a monomial vanishes identically on the power solution, as $(\nabla^2\Phi)\partial_i\partial_j\Phi\,\partial_j\Phi$ does, $s = (D-2)/N$ is unusable without a logarithmic correction. No general treatment of this exception has been written.
- **It is not claimed that "dark matter is unnecessary".** What could be said concerns only **the number of options**.

81-11. The question that stands next

Three items of Phase 4 remain. **4-2 (E(iii)')** adjudicates Chapter 71's conjecture — "the absence of the constraint algebra and the non-locality of the modular Hamiltonian are two faces of the same omission". Now that new 1-15 (K245-K392) has shown that an irregular causal set has a deformation algebra that closes in the same form as GR, the antecedent of the conjecture is broken. Whether the modular Hamiltonian came closer to local on the side where the algebra returned will be decided by **re-running the regular lattice's 0.331 and the causal set's 0.406 (Chapters 58 and 59) side by side at the same operating point**. The existing numbers point the other way. **4-3 (F(i))** takes the "the fluctuation in 4-volume is Poisson, hence $\sqrt{V}$" at the core of everpresent Λ and starts from measuring the fluctuation itself (K383's overdispersion $\sigma_{\text{measured}}/\sigma_{\text{Poisson}} = 1.61$ showed that it cannot be used untested). **4-4 (F(ii))** is $a_0 \simeq cH_0/2\pi$ — running the conclusion of this chapter on an expanding lattice, the one section of Phase 4 carrying "high" risk. The lever of this chapter, $r^2F_r = GM$, does not hold on an expanding background. What remains there is not yet known.

Part XXIV What changed since the ninth edition

> **What this part sets out to establish.** Chapter 73 summarised what had been reached by the ninth edition. This part is the continuation. The verification plan of 2026-08-21 named **three gaps** in this book — the absence of a generator, a measure still put in by hand, and the untested generalisation that "regularity is the culprit for everything". After 217 calibrations from K198 to K399, what has become of the three? **What was filled, what changed shape, and what holes newly opened — counted with the same severity.**

Chapter 82 Were the three gaps filled? ⚠️🆕

— What 217 calibrations since the ninth edition returned

The verification plan of 2026-08-21 (`Research_Plan_Open_Problems_ja.md`) named **three gaps** in the ninth edition of this book. Chapter 73's summary stops there. This chapter is the continuation.

82-1. The three gaps as named

	Gap	How it was put at the time
1	**The generator is absent**	A CA has refoliation **invariance**, but no **algebra that generates** it (Chapter 71). All that was recorded was the guess that this is the same wall where Chapter 68 stopped at link (2), and there it stopped
2	**The measure was put in by hand**	The axioms did not single out one quantum measure (Part XIII). Why $\lvert\cdot\rvert^2$ is $\lvert\cdot\rvert^2$ has not even been made into a question yet
3	**Regularity is the culprit for everything**	The Lorentz obstruction, the K_4 anisotropy, and the fact that the "now" cannot be tilted were all due to regularity, not to discreteness. **But what a discrete world with regularity thrown away recovers, and what it loses, has still not been measured head-on**

Each is judged below, in turn.

82-2. Gap 1 — it was not "absence" but "shortfall" ⚠️

> In a regular CA the deformation algebra is **exactly commutative** (Chapter 74). There is no generator.
> On an irregular causal set there **is** one. Its closure too depends on q^{ab} (Chapters 77-80).
> What is missing is **only the amplitude**.

This is the largest change since the ninth edition. **The nature of the gap has changed.**

	Up to the ninth edition	Tenth edition
generator	nowhere to be found	**there is one** (on irregular discreteness)
q^{ab} dependence of the closure	unmeasured	diagonal and off-diagonal both measured. **The form fits with** $\chi^2/\mathrm{d.o.f.} = 0.54$
symmetry $\varepsilon_A(c) = \varepsilon_D(-c)$	—	machine precision (5.6×10^{-16})
quantitative agreement	—	$R = 0.116 \pm 0.037$. **Only** 12% **of the continuum comes back**

"Absence" has become "shortfall". And the size of the shortfall shows a regularity of its own — $R \simeq 0.0105\,\ell T$ (ℓ the depth of the ripple, T the length of the flow, both measured in link time). It is a fit to three points, and **a hypothesis, not a claim.** It is the first task of the fourth generation and after.

And with "absence" gone, the conjecture of Chapter 71 is left hanging. The conjecture was that the absence of a constraint algebra and the non-locality of the modular Hamiltonian are two faces of one gap. But on the side where the algebra came back — the irregular causal set — the modular Hamiltonian in fact **reaches further** (causal set 0.406 against 0.331 on the regular lattice; Chapters 58 and 59). **A conjecture whose antecedent has broken calls not for a test but for an adjudication at matched operating points.** It is deferred to 4-2 of the fourth generation.

82-3. Gap 2 — filled only halfway ⚠️

Chapter 75 measured the hierarchy of measures and showed numerically that level 2 is isolated. And for the arrow of time, it confirmed that Loschmidt reversal works exactly.

But it does not answer "why $|\cdot|^2$". What it answers is "what lies around $|\cdot|^2$". In particular it **does not so much as touch "why complex numbers"** (stated explicitly in Chapter 75's "What is not being claimed").

Gap 2 has **become a question**. It is not filled.

82-4. Gap 3 — filled, and the price measured ✅

This is the trilemma of Chapter 76.

Discrete, exactly local, Lorentz invariant — **the three cannot hold at once.**

Give up regularity and Lorentz comes back. In exchange **locality breaks logarithmically**. This is one of the strongest statements in this book, and it has **the shape that dies to a single counterexample**. That is healthy.

And Chapter 81 tightened the same "the problem is the shape, not the regularity" one notch further on the rotation-curve side. What blocks a flat rotation curve is neither discreteness nor regularity but **parity** — $\mathbf{F}$ is a vector, so the total number of derivatives D is odd, while flatness demands $D = 2$. The dichotomy of Chapter 70 is strengthened into one carrying a gap $(-1, 1/3)$.

82-5. The count

	count
calibrations (K198-K399, cumulative)	**217**
failures	**113**
— wrong predictions	72
— errors of declaration or design	32
— bugs I introduced myself	9

Failure rate 52%. One time in two is a miss.

Not hiding this number is the method of this book. And looking at the breakdown, **the errors of declaration and design (32) hurt more than the wrong predictions (72)**. A wrong prediction is only being wrong about the world; an error of declaration or design means **having designed an experiment that cannot measure anything**.

Up to the ninth edition (K1-K197) the failures were far fewer. The reason is simple: **back then I was measuring things that are easy to measure.** Since the ninth edition the work has moved into quantities whose target value is unknown, quantities that need an estimate of σ, and quantities where the choice of operating point by itself decides the conclusion.

82-6. Three additions to the method 🆕

The breakdown of the failures changed, so the method changed too. **These are the standard procedure from the fourth generation on.**

	Procedure	Which failure it grew out of
1	**Reconnaissance** — scan the operating points first on the continuum side (or on an exact solution), and only then run the causal set	Chapter 80. An operating point chosen without reconnaissance returned only $\lvert k_c\rvert/\varepsilon = 0.042$
2	**A power calculation before running** — predict the width of the conclusion's interval from the measured σ, and **if it will not settle, do not run**	Chapter 80, F112. Computing **before running** that pinning w to a width of 0.8 would take 5×10^2 core hours, I stood down
3	**Check in advance the range over which the family of models is valid**	Chapter 80. The continuum side itself lay outside the family $\varepsilon = F + k\mu^w$, and the "width 2.93" was not an error bar but **a breakdown of the model**

Better to know before running that it will not settle than to run and only then find out.

82-7. A new basso continuo 💭

Three more have joined the basso continuo of the ninth edition (Appendix D) since.

(D-5) Once a shock stands up, the very notion of a power disappears. Chapter 80 found that $\varepsilon \propto q^{ab} n_a n_b$ (the GR prediction, $w = 1$) holds only while $\max \lvert\nabla f\rvert_q < 1$. Once the ridge breaks the response saturates, and **no power explains the shape**. The caustic of Chapter 79 and Chapter 81's "the real solution vanishes at a finite radius" have the same shape. **Outside the window of linear response, trying to measure an exponent is itself the error.**

(D-6) One's own assumptions pin the operating point. Ever since Chapter 79 I had believed that "A_0 must be several times the layer thickness" and had constrained the operating point accordingly; measured in Chapter 80, **retention did not drop even at 1.1 layers**. For three sections I had been designing inside a constraint that was not needed. **Do not use an unverified constraint as a constraint.**

(D-7) The reachable values come in steps, and the value one wants sits in a gap. $I \geq 2$ in Chapter 33, $s \in \{-1, 1/3, 3/5, \ldots\}$ in Chapter 70, and now $s = (D-2)/N$ in Chapter 81 — the third time. **A degree of freedom that looks continuous is in fact discrete, and the value one wants falls into the gap between its values.** It is the structure that recurs most often in this book, and each time a dichotomy — "then what do we give up?" — has grown out of it.

82-8. The holes that remain ⚠️

Listed honestly.

1. **The origin of R.** $R \simeq 0.0105\,\ell T$ is a fit to three points. Vary ℓ and T independently and knock it down.
2. **Adjudicating the conjecture of Chapter 71.** There is still no comparison at matched operating points.
3. **Everpresent Λ.** The assumption at its core, "the fluctuation of the 4-volume is Poisson", **cannot be used untested** (in K383 of Chapter 80 the variance of the estimator was measured at 1.61 times Poisson — overdispersed). Start from measuring the fluctuation itself.
4. $a_0 \simeq cH_0/2\pi$**.** The lever of Chapter 81, $r^2 F_r = GM$, does not hold on an expanding background. What survives is unknown.
5. **Uniqueness of the measure.** "Why complex numbers".
6. **Quantization.** Part XXII is **classical** from beginning to end: a torus, constant q_{ab}, $2+1$ dimensions.

What this book gained since the ninth edition is **where the generator is**.
What it has not gained is **how big it is**, and into **quantization** it has not taken a single step.

82-9. What is not being claimed ⚠️

- **I do not say that the deformation algebra has been constructed on a causal set.** What was measured is **the behaviour of an estimator**, not the Poisson bracket itself. No phase space, no constraint, no canonical structure has been laid down.
- **I do not say that R approaches 1.** What was written down is only the **computational cost** of the road towards it.
- **The trilemma dies to a single counterexample.** That is healthy.
- **Which side is right about rotation curves has not been decided.** It cannot be. The limitation stated in Chapter 70 stands unchanged.
- **A failure rate of 52% is neither a boast nor a shame.** It is what comes of trying to measure things that are hard to measure, and it is a number that **goes down if one measures only what is easy to measure**.

Part XXV Differences and absolutes — Casimir, Jacobson, and the hydrodynamic limit

Chapter 83 Differences are universal, absolutes are not 💭📚

— What it means for laws of spacetime to arise "automatically" from vacuum energy

In §5-1 this book conceded that a CA does **not** evade the cosmological-constant problem: the lattice stops the divergence but does not make the number small. Parts IX and XIII then measured that necessary conditions for the Einstein equation nonetheless hold in boundary entanglement (Chapter 36; restricted in the thirteenth edition, §87-2), that one link of the chain is cut by regularity (§58-3), and that on an irregular causal set the scaling of the area law appears while its coefficient does not (Chapter 60).

This chapter gathers those results into a single sentence. **Differences of vacuum energy are universal; the absolute value is not.** It then shows that this sentence is formally isomorphic to a structure already familiar from the hydrodynamic limit of CA — "the form of the equation is fixed by conservation and symmetry, the transport coefficients by the collision rule" — and extracts from the places where the isomorphism breaks four quantities to be measured in the fourth period.

The point of departure is a phenomenon this book has not yet treated: the Casimir effect.

83-1. Two things the Casimir effect teaches 📚

The vacuum energy between parallel conducting plates is the difference of two divergent zero-point sums,

$$E_{\mathrm{Cas}}(d) = -\frac{\pi^2 \hbar c}{720}\frac{A}{d^3},$$

and is **independent of the cutoff**. In real materials Lifshitz theory supplies the cutoff through the dielectric function $\varepsilon(\omega)$ (the material becomes transparent above its plasma frequency), and that dependence shows up as finite-conductivity corrections without altering the leading term.

Two things follow.

First. The Casimir force never refers to the **absolute value** of the zero-point energy. As Jaffe (2005) showed, it can be computed as the sum of photon exchanges between charged matter (retarded van der Waals interaction) and vanishes as the coupling $\alpha \to 0$. The Casimir effect is therefore not evidence that vacuum energy is real and gravitates. The cosmological-constant problem of §5-1 is not lightened by invoking it.

Second. No matter how finely one resolves the structure of matter, weak-interaction corrections from the Standard Model never enter. The range of the W/Z, $1/M_W \approx 2 \times 10^{-18}$ m, is more than ten orders of magnitude below the typical Casimir separations (10 nm to $1\ \mu$m), and the contribution of a field of mass m falls as e^{-2md}. Indirect corrections to the material's $\varepsilon(\omega)$ are of the order of atomic parity violation (relative 10^{-11}), buried under the modelling uncertainty of $\varepsilon(\omega)$ (10^{-2}). Conversely, precision Casimir measurements are the tool that **constrains** unknown Yukawa-type forces below the micrometre.

It is the first point that this chapter uses. **Differences universal, absolutes not.** We apply it to three derivations of gravity and to the hydrodynamic limit of CA.

83-2. Three derivations side by side 📚

Among derivations claiming that gravity arises "automatically", there is one this book has already used and one it has not.

(A) Jacobson's thermodynamic derivation (1995). For the local Rindler horizon at an arbitrary point, demand the Clausius relation $\delta Q = T\,dS$ at the Unruh temperature $T = \hbar a/2\pi c$, together with $S = \eta A$. Relating the area change to curvature through the Raychaudhuri equation, the necessary and sufficient condition for this to hold on every local horizon is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{2\pi}{\hbar\eta}T_{\mu\nu}, \qquad \eta = \frac{1}{4G\hbar}.$$

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The inputs are three: Clausius, an area law, and causal structure. η is an input; Λ is an integration constant. Jacobson (1994) identified η with the entanglement entropy across the horizon, $S \simeq c_d A/\varepsilon^{d-2}$, obtaining $G \sim \varepsilon^2$ (Sakharov's induced gravity). **This is a bulk derivation; it needs no boundary theory.**

(B) Derivation from the boundary first law (Faulkner–Van Raamsdonk et al., 2014). Imposing $\delta S_A = \delta\langle K_A\rangle$ for ball-shaped boundary regions of every position, size, and orientation yields, under the RT dictionary, the linearised Einstein equation. This is what Chapters 36, 58, and 60 called the "Jacobson–Van Raamsdonk type" and measured in three steps (area law → locality of the modular Hamiltonian → first law). **This is a boundary derivation; it needs a holographic dictionary.**

(C) The hydrodynamic limit of a CA (Chapman–Enskog). For a lattice gas on a hexagonal lattice (FHP), the local collision rule, the conserved quantities (mass, momentum), and the lattice symmetry group yield

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\big(g(\rho)\,\rho\,\mathbf{u}\mathbf{u}\big) = -\nabla p + \nu\nabla^2(\rho\mathbf{u}).$$

The form is fixed by conservation and symmetry. The viscosity ν is a non-universal quantity computed from the collision rule; $g(\rho)$ is a factor expressing the lattice's breaking of Galilean invariance, absent in the continuum; on a square lattice (HPP) isotropy of the fourth-rank tensor is not guaranteed and the viscous stress is anisotropic.

(A) and (B) are different derivations. So far this book has measured only (B). (A) is closer to a CA in needing no boundary theory, but it requires a lattice counterpart of the Unruh temperature — a temperature proportional to acceleration. We return to this in §83-4.

83-3. The isomorphism — form automatic, coefficients non-universal 💭

Element	(A) Jacobson	(B) Boundary first law (Ch. 36, 58, 60)	(C) Hydrodynamic limit
Fundamental degrees of freedom	Planck-scale field modes	Boundary CA (free-fermion chain)	Cells (alphabet k)
Local rule	Clausius relation on local horizons	First law per region	Transition rule
Conserved quantity	$T_{\mu\nu}$	Boundary T_{ab}	Mass, momentum
Causal structure	Local light cone	Boundary light cone	Dependence cone (Ch. 3)
Cutoff ε	Short-distance cutoff of entanglement	Lattice spacing	Lattice spacing
Fixed automatically	Form of the Einstein equation	Form of linearised Einstein (Ch. 36: necessary conditions only ⚠️; §87-2)	Form of the continuum equation
Not fixed	$G \sim \varepsilon^2$, Λ	$G = 3/(2c)$ was **defined** via RT (Ch. 35); coefficient (Ch. 60)	ν, equilibrium pressure
Symmetry breaking	Lorentz (substrate-dependent)	Lorentz on the regular lattice (Ch. 24)	$g(\rho)$, lattice anisotropy
Universality of differences	Casimir energy	—	Critical Casimir amplitude
Non-universality of absolutes	$\rho_{\text{vac}} \sim \varepsilon^{-4}$ (§5-1)	same	Bulk pressure $O(1)$ (lattice units)

What the three columns share is the structure of the renormalisation group. **At each order of the gradient expansion, the form of the permitted terms is fixed by symmetry and their coefficients by the microscopic rule.** (A) yields the Einstein equation because at first order in curvature the only terms permitted by locality and general covariance are $G_{\mu\nu}$ and $g_{\mu\nu}$ (Lovelock, §58-3). (C) yields Navier–Stokes because at second order in gradients the only terms permitted by rotational symmetry and conservation are the pressure gradient and the viscous term.

> ”The law arises automatically” means, precisely, ”the **form** of the lowest-order terms permitted by symmetry and conservation arises automatically.”

Under this reading the three columns make the same claim, and Chapter 60's "the area law appears as a scaling but not as a coefficient" is the same sentence as "(C) yields Navier–Stokes but not ν."

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83-3-1. The critical Casimir effect — the lattice version of the cosmological-constant problem 📚

The lattice counterpart of the Casimir effect is the critical Casimir effect of Fisher and de Gennes (1978). The free energy per unit area of a lattice system of thickness L expands as

$$f(L) = f_{\text{bulk}}\, L + 2 f_{\text{surf}} + \frac{\Delta}{L^{d-1}} + \cdots,$$

and **when the system is at a critical point** (divergent correlation length, i.e. the emergent field is massless) the amplitude Δ is universal, independent of lattice details. In two dimensions it is fixed by the central charge — $-\pi c/6$ for a strip with periodic boundaries. Meanwhile f_{bulk} and f_{surf} are non-universal quantities of order one in lattice units.

This is the lattice version of the first point of §83-1. The structure by which a vacuum energy density $\sim \varepsilon^{-4}$ produces the cosmological-constant problem is the structure by which the bulk pressure of a lattice gas is $O(1)$ in lattice units: in both, **the emergent continuum theory inherits the non-universal constant term of the microscopic rule.** On the CA side the only ways to remove it are to impose a symmetry on the rule (particle–hole symmetry cancelling part of the bulk pressure) or to fine-tune. That the known options for making Λ small are supersymmetry, fine-tuning, or anthropic selection is a one-to-one correspondence.

The cosmological-constant problem of § 5-1 can be restated on the CA side as "bulk pressure is $O(1)$ in lattice units."

One consequence follows. **A CA substrate produces a universal Casimir-type difference only when the substrate is at a critical point.** The masslessness of the photon and the graviton, in the CA picture, demands that the substrate sit at (or self-organise to) criticality. The free-fermion chain of this book is critical at half filling ($c = 1/2$, Chapter 35), so the condition is met; §83-6 (a) measures it directly.

83-3-2. A lattice expression for G 💭

Reading the area-law coefficient of (A) in CA terms, the upper bound on the information that can cross a boundary in one step is the number of boundary cells $\times \ln k$. Identifying this with $1/4G$ gives, in lattice units,

$$\frac{1}{4G} = \frac{\ln k}{\varepsilon^{d-2}},$$

so that in $d = 4$, $G \propto \varepsilon^2/\ln k$, consistent with Sakharov; for Life, $\ln 2$. This does not predict G; it pushes G onto the two microscopic parameters ε and k. But the target is explicit, one step more concrete than continuum induced gravity, and it reaches G by a **different route** from Chapter 35's $G = 3/(2c)$ (defined via RT). Whether the two agree in the same model is the test of §83-6 (b).

83-4. Four places where the isomorphism breaks 💭

83-4-1. Is entropy a count of states, or a flux?

(A) presupposes $S \propto A$. In a generic CA, block entropy scales with volume — the violation of the Bekenstein bound that 't Hooft raised in arguing for holography, and which in this book is the bulk CA of Chapter 35 (10^{60} times the Bekenstein bound). On the causal set of Chapter 60 one found $S \propto N^{0.719}$, an intermediate power, neither area nor volume.

But the dS in (A) is not the number of horizon states; it is the **flux across the horizon**. A CA has an exact dependence cone, so the information the exterior of a region can obtain about its interior in one step is bounded by the number of boundary cells $\times \ln k$, and this "flux entropy" obeys an area law. **§72-3, which measured that "the growth rate of correlations is set by the boundary", is exactly this quantity.** This book has therefore already verified, on a Clifford CA, that the area law is restored once entropy is read as flux rather than as a count of states.

What remains is the T in $T\,dS$ — the lattice counterpart of the Unruh temperature, proportional to acceleration. The acoustic horizon of Chapter 69 returned $T = \kappa/2\pi$, but that was a position-dependent rule with flow inserted by hand, not the local Rindler horizon of an arbitrary point. §83-6 (d) measures it head-on.

83-4-2. Lorentz — $g(\rho)$ is dimension four, anisotropy is dimension six

Chapter 24 divided Lorentz violation into two kinds: dimension six (anisotropy of the dispersion relation, suppressed as $(ka)^2/36$, harmless) and dimension four (species-dependent limiting speeds, unmoved by coarse-graining, the real problem).

The hydrodynamic limit (C) exhibits both kinds **in exact correspondence**.

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Chapter 24	(C) lattice gas	Suppression	Remedy
Dimension 4: mismatched limiting speeds	$g(\rho) \neq 1$ (Galilean violation in the convective term)	none	tune the density to a specific value (= set by hand)
Dimension 6: dispersion anisotropy	anisotropic viscous stress of HPP (fourth-rank tensor)	relative, since second order in gradients	hexagonal lattice (enlarge the symmetry group)

$g(\rho)$ multiplies the convective term, an $O(1)$ factor at lowest order that survives coarse-graining. The lattice-gas remedies — tune the density, or extend to multi-speed models — are the same prescription as §24-2's "set by hand, species by species." The HPP anisotropy, by contrast, multiplies the viscous term, second order in gradients; it disappears when a hexagonal lattice isotropises the fourth-rank tensor, and is in any case relatively suppressed at long wavelength. It sits in the same tier as what §24-1 judged "harmless."

> Forty years of lattice-gas experience reproduce Chapter 24's dichotomy exactly. Only dimension four is serious.

The Dirac walk of this book also has a species-dependent limiting speed **generated automatically by the rule**. From $\cos\omega = \cos k \cos\theta$ the maximum group velocity is exactly $\cos\theta$ (reached at $k = \pi/2$), so species of mass angle θ differ in limiting speed by $1 - \theta^2/2$. In form this is a dimension-four violation, but its coefficient is suppressed by $(a/\lambda_C)^2 - 10^{-45}$ for an electron on a Planck lattice — and harmless. The "real problem" of §24-2 is not this automatic difference but the failure of **hand-assigned coupling differences** to flow (see the proposed §6-7 addendum).

§24-3 and Chapter 76, "randomness buys Lorentz invariance", also have a counterpart 📚. Direct-simulation Monte Carlo (DSMC), which leaves the lattice and moves particles in continuous coordinates, restores Galilean invariance. The price is strict locality (collision partners are chosen stochastically) — the fluid version of Chapter 76's trilemma "discrete, strictly local, Lorentz invariant: not all three."

83-4-3. Reversibility — the Clausius equality holds only in second-order form

(A) is thermodynamic and presupposes that thermodynamics arises as the coarse-graining of a microscopically reversible dynamics. Chapter 15 counted that Life's image-size loss is 0.0955 bits per cell per step (the uniform-input loss is 0.2615; §87-4), and that the second-order form, a different reversible dynamics, discards exactly 0. In an irreversible rule, Landauer-type dissipation accompanies the information flux across a boundary and the equality $\delta Q = T\,dS$ becomes $\delta Q \leq T\,dS$. As Jacobson later discussed, under the inequality one obtains not the Einstein equation but an equation with non-equilibrium correction terms.

Any attempt to reproduce (A) on a CA with equality must therefore use a reversible rule — **second-order Life, Q2R, Critters**. A reproduction on plain Life is meaningful only as a model of a "dissipative modified Einstein equation." This is the gravitational consequence of Chapter 15's conclusion that erasure arises not from the rule but from the definition of the state.

83-4-4. Quantumness — an area law alone is not entanglement

Finally, the area laws of (A) and (B) concern quantum entanglement entropy, while the block entropy of (C) is classical correlation entropy. Both can yield an area law, but the latter carries no Bell-type correlations (Chapters 20, 49). As Chapter 37 and `Holographic_Gravity_From_Constraints_ja.md` (2026-09-13) point out, multiplying free fermions produces neither sparseness nor large-C_T factorisation. **Obtaining an area law is not reproducing entanglement.** Of the four breaks, this is the only one that no choice of rule or reinterpretation closes.

83-5. What this chapter does not claim ⚠️

- It does not claim to derive the Einstein equation from a CA. The claim is only that the structure "form automatic, coefficients non-universal" is common to the three derivations.
- The universality of the critical Casimir amplitude is a literature result (📚), not measured in this book's models.
- §83-3-2's $G = \ln k/\varepsilon^{d-2}$ is an identification, not a derivation; its consistency with Chapter 35's $G = 3/(2c)$ is untested.
- The table of §83-4-2 juxtaposes Chapter 24's numbers with the lattice-gas literature; both have not been measured in one model.
- No lattice counterpart of the Unruh temperature has been defined. §83-6 (d) searches for one; it does not presuppose one exists.

83-6. Fourth-period tasks — four measurements

All are executable with this book's existing tools (Appendix B, `CADoc/v2`). None carries 🆕 until executed.

(a) Critical Casimir amplitude. On the free-fermion chain of Chapter 35 (half filling, $c = 1/2$), measure the ground-state energy $E_0(L)$ of a periodic chain of length L and show that the amplitude in $E_0(L) - L\,e_{\text{bulk}} \to -\pi c\, v_F/(6L)$ is invariant under deformations of the hopping (changing μ, adding next-nearest-neighbour terms) while e_{bulk} changes. Show also that off criticality the amplitude vanishes as $e^{-L/\xi}$. This directly tests §83-3-1, and "e_{bulk} is non-universal" is the lattice version of §5-1's cosmological-constant problem. As a non-free check, measure the same quantity at the DP critical point of the Domany–Kinzel probabilistic CA.

(b) State-count entropy versus flux entropy. For Life, second-order Life, and Q2R, measure the state entropy of a block of side ℓ and the one-step change of the conditional entropy across its boundary; show that the former scales as ℓ^2 and the latter as ℓ. This extends §72-3's Clifford result to non-Clifford rules. Insert the measured coefficient $\ln k$ into §83-3-2 and compare with Chapter 35's $G = 3/(2c)$.

> ⚠️ **Addition (twelfth edition).** The classical $S_l = s_0 l + C(t)$ has been partly measured, for the one-dimensional rules 30 and 110 (K409, §86-4). A non-Clifford quantum circuit was checked at $t = 1$ on 12+12 qubits (K410). The ℓ^2 versus ℓ scaling for Life, second-order Life and Q2R, and the conditional entropy of the flux, remain unmeasured.

(c) Fourth-rank isotropy and the Chapter 24 correspondence. Measure the angular dependence of stress in HPP and FHP and show that the anisotropy is suppressed quadratically in wavenumber (dimension-six type). Simultaneously introduce two speed species and show that the $g(\rho)$ mismatch does not flow under coarse-graining (dimension-four type). Fill the table of §83-4-2 within one model.

(d) Local Clausius relation. In an FHP lattice gas, measure the ratio of the momentum flux crossing a virtual boundary moving at constant velocity v to the information flux of (b), and examine whether the ratio is linear in v. If so, the slope is a candidate lattice Unruh temperature. If not, that is evidence that the open problem of §83-4-1 is a genuine obstacle. **Either result makes a chapter.**

83-7. Where this chapter sits 💭

Parts IX and XIII measured route (B) and found one link cut by regularity. This chapter translated route (A) into CA language and, through the isomorphism with (C), showed that the cut lies in the same two places as for (B) — the non-universality of coefficients and the dimension-four Lorentz violation.

The wish that laws of spacetime arise "automatically" from vacuum energy is already granted as to form (Chapters 36, 60). What is not granted is the coefficient — which is what the Casimir effect had been saying from the start.

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Part XXVI Why is gravity weak? — Dimension fixes the exponent, coupling fixes the coefficient

Chapter 84 $d = 3$ fixes the inverse square, not the smallness of G 💭📚

— Is the weakness of gravity a question of the dimension of space, or of coupling?

Chapter 25 showed that $d = 3$ is demanded three times independently; Chapter 35 showed that placing the rule on the boundary makes the 10^{61} problem disappear. Between those two chapters lies a question this book has not yet faced directly. **Why is gravity thirty-six orders of magnitude weaker than electromagnetism?** And is there anything a CA can say about it?

The answer first. Dimension fixes the **exponent** of the distance dependence of a force: $d = 3$ gives $1/r^2$. But the smallness of the coefficient G does not follow from dimension. The weakness of gravity is not a question of dimension but of **coupling**, and coupling sits, in the language of Chapter 83, on the side of "non-universal coefficients". This chapter pursues the question by examining a proposal that starts from the exclusion principle, and finally connects it to the holographic formulation (in the form corrected by items 1 and 3 of Chapter 85).

84-1. One premise corrected 📚

The "weak interaction" is not a long-range force obeying an inverse square. Because the W and Z are heavy, the static exchange interaction is of Yukawa type,

$$V(r) \propto \frac{e^{-r/\lambda}}{r}, \qquad \lambda = \frac{\hbar}{Mc} \simeq 2 \times 10^{-18}\ \mathrm{m}.$$

The mass term of the Klein–Gordon equation of §6-7 produces this decay length — the same reason §83-1 gave for weak-interaction corrections never entering the Casimir effect. What this chapter compares is gravity and electromagnetism, both long-range forces with massless mediators.

84-2. Gauss's law on the lattice — dimension fixes the exponent 📚

Suppose a conserved flux is distributed over cells whose number grows with distance. In d spatial dimensions the number of cells on the analogue of a sphere of radius r scales as r^{d-1}, so

$$\boxed{F(r) \propto \underbrace{C}_{\text{coupling}} \cdot \underbrace{\frac{1}{r^{d-1}}}_{\text{dimension}}}$$

The premises are three — massless propagation, isotropy, flux conservation. Not every nearest-neighbour interaction takes this form; the lattice Green functions of §25-1 (linear in $d = 1$, $\log r$ in $d = 2$, $1/r$ in $d = 3$) are precisely measurements under these three premises.

The essential point is that **dimension fixes the exponent, coupling fixes the coefficient C.** In the same $d = 3$ the coefficient is free. For two protons

$$\frac{F_G}{F_{\mathrm{EM}}} = \frac{G m_p^2}{e^2/4\pi\varepsilon_0} \simeq 8.1 \times 10^{-37},$$

with no distance left in the ratio since both are inverse-square. For two electrons the ratio is $(m_e/m_p)^2$ times this, 2.4×10^{-43}.

> "The weakness of gravity" is not one number. It depends on which particles are compared.

This has the same shape as Chapter 26, "The question of mass ratios". Asking why G is small is asking about the dimensionless ratio Gm^2/e^2; since a mass enters, it cannot be separated from the mass-ratio problem.

84-3. Dilution by extra dimensions 📚

The only route that returns the smallness of the coefficient to dimension is to let gravity alone spread into n extra dimensions:

$$G_4 \sim \frac{G_{4+n}}{V_n}.$$

The volume V_n of the extra dimensions dilutes the effective gravity; below the size of the extra dimensions $F \propto r^{-(2+n)}$, above it r^{-2} returns (Arkani-Hamed–Dimopoulos–Dvali). **Weakness alone does not fix** n; the volume and shape are needed.

In this book the route corresponds to Chapter 33, "Hidden dimensions". Chapter 33 examined hidden dimensions as a mechanism for the mass hierarchy and found its invariant confined to $I \geq 2$, unable to reach the $32/27$ of §32-5. **Whether the dilution of gravity and the generation of mass ratios can come from the same hidden dimension is a question that can be asked within the framework of Chapter 33** — but the book has not asked it. Recorded as a fourth-period candidate.

84-4. The exclusion principle does not become gravity ⚠️

Consider the proposal that "a restriction on which states can exist appears as a force." Part of the starting point is right.

The Pauli principle says that identical fermions cannot occupy the same one-particle state — not that they cannot be in the same place. Under this restriction, packing particles into a small region forces the filling of high-momentum states, and for a zero-temperature non-relativistic ideal spin-$\frac{1}{2}$ Fermi gas the degeneracy pressure is

$$P = \frac{\hbar^2}{5m}(3\pi^2)^{2/3} n^{5/3}.$$

This is a real instance of **pressure arising from a restriction on states without introducing a new mediating particle.** Chapter 8's "E/n from -4.04 to -3.33, Pauli pressure" is its CA version, and Chapter 12, "Constraints force interactions", showed that under the local constraints of that model there are zero non-trivial dynamics expressible as bilinears (not a general result; §87-6).

But this cannot be taken for gravity.

Pressure from exclusion	What gravity requires
resists compression (repulsive)	attractive between ordinary matter
depends on occupied states, density, species	the same free fall regardless of composition
originates in Fermi statistics	acts on light, bosons, binding energy
produces no universal $1/r$ in vacuum	acts between distant bodies

Composition dependence in particular is severe. MICROSCOPE measured the difference in free fall between titanium and platinum to 10^{-15} and saw no violation of the equivalence principle 📚.

Constraints produce interactions (Chapter 12). But **why that interaction should be gravity** does not follow from the constraint.

Chapter 12 and this section do not conflict. Chapter 12 showed the existence of an interaction; this section concerns its kind.

84-5. Deforming the background — the minimal model that yields attraction 💭

Widen the proposal by one step.

> The presence of a body changes which states the background may occupy, and the background has lower energy when two bodies are placed together.

In this form attraction can be produced. The quantum example is the Casimir effect (§83-1): boundary conditions change the state of a field and a force arises between bodies. But that is an electromagnetic effect depending on material and shape, not gravity.

Write it as a nearest-neighbour cell model in minimal form. **What follows is a hypothetical model for studying the mechanism, not a theory derived from the exclusion principle.** Let χ be the deformation of the background state in each cell, with an energy cost when neighbouring values differ. The long-wavelength expression is

$$E[\chi;\rho] = \int d^3x \Big[\frac{K}{2}|\nabla\chi|^2 + g\rho\chi\Big],$$

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with ρ the mass density of the bodies, $K > 0$ the stiffness of the background against deformation, g the strength with which bodies act on the background. On the lattice the gradient term is the squared difference between neighbouring cells. Fixing the bodies and minimising the background energy gives $K\nabla^2\chi = g\rho$; with the deformation vanishing at infinity, a point body produces a $1/r$ deformation, and the energy between two bodies, self-energies removed, is

$$\boxed{V_{\text{eff}}(r) = -\frac{g^2}{4\pi K}\frac{m_1 m_2}{r}, \qquad G_{\text{eff}} = \frac{g^2}{4\pi K}}$$

An inverse-square attraction results.

What the model shows is that **even with space kept three-dimensional, a weak long-range attraction can be made if the coupling to the background is small or the background is hard to deform.** The physical combination is g^2/K; this is not a matter of making the force look weak by changing units.

Here is where "increasing the number of interacting elements" can enter. If many internal degrees of freedom participate together in the deformation of the background and their energy costs add,

$$K_{\text{eff}} \propto N \quad \Longrightarrow \quad G_{\text{eff}} \propto \frac{1}{N}$$

can result. **Only if g does not grow at the same time.** Adding degrees of freedom does not by itself dilute. And this N is not the number of extra spatial dimensions but **the number of internal states supporting one collective deformation**. The dimension of a quantum state space cannot be read as "the number of directions a force spreads into".

Three defects of the model, stated first.

1. If a mass term $\frac{\mu^2}{2}\chi^2$ arises in the background, the force becomes a short-range Yukawa force (the reverse of 84-1).
2. If g differs between species, the equivalence principle is violated; the composition dependence of 84-4 returns.
3. A single scalar variable produces neither gravitational waves nor the bending of light (Einstein's $2\times$ Newton). Tensor structure is needed.

Call these the four properties.

84-6. The holographic formulation 📚

There is a route that ties the four properties to the same quantum structure on a lower-dimensional boundary. We summarise the organisation of `Holographic_Gravity_From_Constraints_ja.md`, in the form corrected by items 1 and 3 of Chapter 85.

The target correspondence is a boundary of two spatial dimensions plus time and a bulk of three spatial dimensions plus time — standardly $\text{AdS}_4/\text{CFT}_3$. The one-dimensional chain of Chapter 35 has one spatial dimension fewer, and the G_3 obtained there is not G_4 (§85-1). The added spatial direction z relates to the coarse-graining scale on the boundary, but not every coarse-graining circuit produces a smooth space.

The central boundary datum is the conserved energy–momentum tensor T_{ab} and its correlation functions. Assuming a unitary CFT_3, define the normalisation C_T of stress correlations by

$$\langle T_{ab}(x) T_{cd}(0)\rangle = C_T \frac{\mathcal{I}_{ab,cd}(x)}{|x|^6}.$$

Wherever a weakly curved Einstein-type bulk exists, with a convention-dependent positive number a_T,

$$\boxed{\frac{G_4}{L^2} = \frac{a_T}{C_T}}$$

In physical units the left side is ℓ_P^2/L^2. **A small gravitational coupling at scales comparable to L corresponds to a large C_T.** This is the concrete form of the g^2/K of 84-5. The coefficient of the quadratic action for a fixed geometric deformation h is proportional to C_T; normalising $h_{\text{can}} \sim \sqrt{C_T}\, h$, the vertex carries $C_T^{-1/2}$ and the exchange amplitude C_T^{-1}:

$$K_{\text{eff}} \sim C_T, \qquad \left(g^2/K\right)_{\text{dimensionless}} \sim C_T^{-1}.$$

The four properties attach to the boundary as follows.

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Property sought	Boundary condition	Bulk consequence and limitation
Long range	exact conservation of T_{ab}; a suitable critical continuum limit	under the dictionary a conserved T_{ab} corresponds to a massless spin-2 field; but long range in AdS is not the flat-space $1/r$
Universal coupling	Ward identities for one and the same total T_{ab}	coupling follows energy–momentum regardless of species; $\lambda_{OO\hat{O}} \propto \Delta_O/\sqrt{C_T}$ — "the same law", not "the same number"
Tensor structure	a conserved spin-2 operator, its correlators, control of higher-spin excitations	a candidate metric fluctuation h_{AB}; Einstein-type interactions need conditions beyond the two-point function
Small coupling	$C_T \gg 1$ and large-N factorisation	vertex $C_T^{-1/2}$, exchange C_T^{-1}

This table must not be read as "a conservation law yields gravity." A conserved T_{ab} exists in free field theory too. An Einstein-type local bulk needs large-N factorisation, a sparse spectrum of light single-particle operators, and a large gap to higher spin (Heemskerk–Penedones–Polchinski–Sully). **This is the identity of Chapter 37's "sparseness wall" restated in boundary language** — the obstacle is not the total number of degrees of freedom but the selection of the collective ones that survive as gravity (§85-3). Placing N independent free fermions side by side multiplies C_T by N, but each copy keeps its own conserved T and many higher-spin conserved currents remain, so the conditions are not met.

The contact with entanglement is the second variation of relative entropy. When the dictionary holds, its Hessian (the quantum Fisher metric) corresponds to the canonical energy of the gravitational perturbation. But, as retracted in §85-2, **positivity of the second variation alone does not establish the dictionary**. The accurate statement is not "much entanglement implies weak gravity" but "for the family of states encoded as an appropriate geometry, the normalisation of the response fixes a weak gravitational coupling."

84-7. The ratio of gravity to electromagnetism is a separate question 📚

A small G_4/L^2 does not by itself explain the extreme weakness of gravity relative to electromagnetism. With C_J the two-point coefficient of the boundary conserved current, the standard correspondence with an AdS_4 Maxwell field gives $e_4^2 \sim C_J^{-1}$ and $G_4/L^2 \sim C_T^{-1}$, so that for particles of mass m and charge qe_4

$$\boxed{\frac{|F_G|}{|F_{\mathrm{EM}}|} \sim \frac{C_J}{C_T}\,\frac{(mL)^2}{q^2}}$$

Making C_T large does not explain the observed ratio if C_J and mL change with it. The normalisations of current and stress and the matter spectrum must be obtained from the same boundary theory. The statement of 84-2 that "the weakness of gravity is not one number" here takes concrete form.

Accordingly a correction box is placed in Chapter 35. "The 10^{61} problem disappears" is a statement about the ratio to the Bekenstein bound, not a derivation of G/e^2; and since cutting cells at the Planck length already uses G, explaining G from the cell count would be circular (§85-1).

84-8. A concrete example, and its limits 📚

A concrete example in which a lower-dimensional quantum theory coexists with a four-dimensional gravitational sector is ABJM theory (the $2+1$-dimensional $U(N)_k \times U(N)_{-k}$ superconformal Chern–Simons–matter theory, dual at large N to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$). It exhibits constraints, strong quantum correlations, and a gravitational description from lower dimensions together. But it contains an internal space, supersymmetry, and many fields; it is not the finite-state, finite-neighbourhood CA of this book. Without a separation between the internal space and the AdS radius, a wide four-dimensional Newtonian regime — "internal dimensions invisible, curvature negligible" — is not automatic for a localised source. There are methods treating the ABJM partition function as an auxiliary Fermi gas, but that Fermi statistics must not be reread as an exclusion principle causing gravity — the caution of 84-4 again.

84-9. What this chapter does not claim ⚠️

- Gravity does not come from the exclusion principle (84-4). Chapter 12's "constraints force interactions" is a theorem about the existence of interactions and says nothing about their kind.

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- $G_{\text{eff}} = g^2/4\pi K$ is a hypothetical model. It reproduces the inverse square and attraction but guarantees neither the equivalence principle nor tensor structure nor long range.
- $G_4/L^2 = a_T/C_T$ is a relation under a holographic dictionary controlled in AdS (negative cosmological constant), not a definition that holds for an unknown CA. Application to the real expanding universe, a positive cosmological constant, and the Standard Model are separate problems.
- The four properties have not been derived simultaneously from this book's free-fermion CA.
- This chapter contains no calculation.

84-10. Fourth-period tasks — not executed

The closing table of `Holographic_Gravity_From_Constraints_ja.md` is placed as is. **None of it has been executed.**

Object of study	Observations and calculations required	What is not claimed at this stage
A boundary quantum model in two spatial dimensions	define the state algebra, local update, conserved quantities, ground state, criticality	not a derivation of quantum theory from a classical CA
Energy–momentum	conservation of T_{ab} at long wavelength, isotropic relativistic scaling, measurement of C_T	a conserved T alone is not gravity
Selection of excitations	light single-particle operators, extra conserved spin-2, higher-spin gap	large C_T alone is not an Einstein bulk
Encoding and region consistency	reconstruction of the same low-energy operator from several regions; consistency under overlapping regions	one entropy curve does not fix the geometry
Area–stress dictionary	whether entropies of several shapes and stress correlators give the same G_4/L^2	defining G via RT alone is not independent verification
Dynamical gravity	tensor equations, retarded response, and canonical energy of bulk perturbations reconstructed from the boundary, computed independently	the first law and a positive second-order response do not pass (§85-2)
Universality	Ward relations of $OO\hat{T}$ for several matter operators	two particles of one kind do not test the equivalence principle
Hierarchy of gravity and the other forces	C_T, C_J, and the spectrum fixing matter masses in the same model	large N alone does not explain ratios such as 10^{-36}
Hidden dimensions and dilution (84-3)	whether, in the superlattice of Chapter 33, dilution of G by V_n and generation of mass ratios come from one mechanism	weakness alone does not fix n

Even starting from nearest-neighbour cells, the coarse-grained effective theory can develop distant couplings and higher-order terms. Rather than demanding every continuous symmetry exactly on a finite lattice, **show in which window of scales the errors are controlled.** An inverse-square law in three spatial dimensions requires a region larger than the short-distance limit of the effective theory and smaller than the background curvature radius.

84-11. Where this chapter sits 💭

Dimension fixes the exponent (Chapter 25, 84-2). Coupling fixes the coefficient (84-5, 84-6).
The weakness of gravity is a question of the coefficient, on the "non-universal" side of Chapter 83.
A CA reaches that coefficient only when C_T and C_J are measured in the same model.

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Part XXVII External verification of the tenth edition — six corrections

Chapter 85 A ✅ for the form was never a ✅ for the coefficient ⚠️🆕

— What the verification received after the tenth edition returned

Chapter 82 recorded how the tenth edition filled the three gaps named against the ninth. This chapter is its sequel: it records **the verification the tenth edition itself received**. It arrived as two documents — `Holographic_Gravity_From_Constraints_ja.md` (2026-09-13, checking Chapters 35–37 and 58–60 against steps 32–40) and a study of the Klein–Gordon equation (same day, checking Chapters 6 and 82). Neither re-ran this book's numerical experiments; both are **cross-checks of equations and claims**.

The cross-check returned six corrections: one retraction, three changed verdicts, one inverted null result, and one restriction of wording (two, counting the change to the Preface that the retraction forces). Three of the six fall in Part IX, "Gravity and holography". **This book's ✅ marks on gravity were ✅ marks for the form of the equation, never for its coefficient.** That is the one line of this chapter.

85-1. Correction 1 — H^2 was called AdS$_2$; $G = 3/(2c)$ is G_3 ⚠️

Chapter 35 tied the free-fermion chain's $S(\ell) = (c/3)\ln \ell$ to a geodesic length through Ryu–Takayanagi, obtaining $G = 3/(2c)$ and, from the measured $c = 0.483$, $G = 3.109$ (exact 3). The metric used there,

$$ds^2 = L^2 \frac{dz^2 + dx^2}{z^2},$$

is the **hyperbolic plane** H^2, the **constant-time slice** of AdS$_3$ spacetime. It is not the Lorentzian spacetime AdS$_2$. The RT relation

$$S = \frac{\mathcal{L}_{\text{geo}}}{4G}, \qquad \mathcal{L}_{\text{geo}} = 2L \ln \frac{\ell}{\varepsilon} \implies c = \frac{3L}{2G_3}$$

is therefore Brown–Henneaux's $c = 3L/2G_3$, and what emerges is the Newton constant of **three spacetime dimensions**, G_3. Chapter 35 wrote it as an unsubscripted G, next to the Part IX heading "The 10^{61} problem disappears", so that it reads as if the four-dimensional G_4 had been reached. It had not.

Two further points. $c \simeq 1/2$ is not the large-central-charge regime in which classical geometry is controlled. And pure Einstein gravity in three spacetime dimensions has no locally propagating graviton, unlike four. The **geodesic** part of Chapter 36, "Emergent geodesics satisfy Einstein's equations", is correctly measured on H^2, but **there is no degree of freedom in $2+1$ dimensions that could be called gravity in the first place**.

	Tenth edition	Eleventh edition
Name of the metric	AdS$_2$	H^2 (constant-time slice of AdS$_3$)
Subscript on G	none	G_3
Scope of "the 10^{61} problem disappears"	weakness of gravity	**counting of information** (volume law → area law); the ratio G/e^2 was not derived

$$\boxed{\text{Chapter 35's conclusion — bulk quantities can be read from the boundary — stands. What was read is } G_3, \text{ not } G_4.}$$

One caution is added to the opening table of Chapter 35. Cutting cells at the Planck length already uses G; "explaining" G from that cell count is circular. What Chapter 35 showed is the **ratio** to the Bekenstein bound (10^{60} for bulk, $1/\pi$ for boundary), not the value of G.

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85-2. Correction 2 — "The second-order Einstein equation also holds" is retracted ❌🆕

§36-2 measured that the relative entropy $S_{\mathrm{rel}} = \Delta\langle H_A\rangle - \Delta S \geq 0$ rises as $\lambda^{1.9934}$ and is positive for every perturbation, both signs, all shapes, and concluded: "a positive-definite quadratic form (the quantum Fisher metric), which is the positivity of the bulk canonical energy, i.e. the second-order Einstein equation." The limitations section of step37 stated explicitly that the bulk canonical energy had not been computed.

This conclusion is retracted. The reason is that a counterexample can be written analytically.

Take independent pairs of qubits, each in

$$|\psi_p\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle, \qquad 0 < p < 1.$$

The reduced state of one qubit is $\sigma = \mathrm{diag}(p, 1-p)$ with modular Hamiltonian $K = -\log\sigma$. Changing $p \to p+\epsilon$, at first order

$$\delta S = \epsilon \log\frac{1-p}{p} = \delta\langle K\rangle$$

(the first law), and at second order

$$D(\rho_{p+\epsilon}\|\rho_p) = \frac{\epsilon^2}{2p(1-p)} + O(\epsilon^3) > 0$$

(a positive-definite quadratic form). **No propagation and no spatial geometry connect the pairs.** Yet both properties that Chapter 36 took as evidence for "the Einstein equation" hold.

Recomputed in this session at $p = 0.3$ (K400–K402):

ϵ	$\Delta S/\Delta\langle K\rangle$	D/ϵ^2
10^{-2}	0.972074	2.366162
10^{-3}	0.997192	2.379444
10^{-4}	0.999719	2.380801
analytic ($\epsilon \to 0$)	1	$1/[2p(1-p)] = 2.380952$

All digits agree with the external document's table.

The attribution, made precise: the first law $\delta S = \delta\langle K\rangle$ is the **identity** that the first variation of $S(\rho)$ about $\rho = \sigma + \delta\rho$ is always $\mathrm{tr}(\delta\rho\, K)$; non-negativity and quadratic onset of relative entropy are Klein's inequality. Both hold for **any family of quantum states**. Geometry is implied only when these are made **consistent, for every region, position and orientation, as area changes of one and the same spacetime** (Chapter 36 tested one family of intervals; Chapter 60's monotonicity test comes a step closer, but neither placed an independent area–stress dictionary).

	Tenth edition	Eleventh edition
Heading of §36-2	The second-order Einstein equation also holds ✅	The second-order response is positive — a necessary condition ⚠️
Preface §序-1, "even the Einstein equation to second order emerges automatically"	—	changed to "the form emerges at linear order (Chapter 36); at second order only a necessary condition"
Appendix C-1 retractions	five	**six**

The first law and a positive second-order response hold for two qubits with no geometry. **By themselves they do not test the Einstein equation.**

To test the second-order Einstein equation one must **construct** the bulk metric and matter fields corresponding to a boundary-defined perturbation and compute **independently** the value of the canonical energy and the residual of the tensor equation. This goes into the fourth-period table of §84-9.

> ❌**Addition (thirteenth edition).** This correction stopped halfway. The same counterexample applies to "the form appears at linear order". §87-2.

85-3. Correction 3 — "The sparseness wall ❌" was not a test ⚠️

Chapter 37 wrote that three attempts hit the same wall: adding interactions does not bring mutual information to zero; raising the central charge c only multiplies mutual information by **exactly** n (I/n agreeing to six digits); sparseness fails as $e^{c/2}$.

The second attempt **was not a test at all**. Placing n free-fermion chains side by side, independently, multiplies mutual information by n additively. That is the additivity of free fields, unrelated to holographic "large C_T". Independent copies give $C_T^{\text{total}} = nC_T^{(1)}$, but each copy keeps its own conserved energy–momentum tensor; they are **not bound into one boundary theory**. The RT-type transition is a phenomenon of the **leading term** when C_T is raised in a theory whose interactions bind everything into a single T_{ab}.

The first attempt also carries a qualification. That mutual information does not vanish exactly at finite degrees of freedom is not a refutation of holography. The classical RT switch concerns the leading large-C_T term; an $O(1)$ mutual information from bulk quantum-field correlations **remains, as the dictionary says it should**.

The verdict of Chapter 37 therefore changes.

	Tenth edition	Eleventh edition
Verdict	❌ sparseness wall	⚠️ scaling of the leading term I/C_T **unmeasured**
Meaning of "I/n to six digits"	evidence of the wall	additivity of independent copies; not a test
Surviving claim	being free fermions is the obstacle	**stands**. The obstacle is not the total number of degrees of freedom but the **selection** of the collective ones that survive as gravity

Chapter 37's box, "sparseness requires a structure in which gauge constraints thin out states", reads on the boundary side as "remove degrees of freedom that are not independently physical, bind them into one conserved T_{ab}, and lift the unwanted light higher-spin excitations." **The verdict changes; the direction does not.** But there is no theorem that constraints guarantee sparseness; each candidate model's spectrum must be measured.

85-4. Correction 4 — restrict the scope of "no additional assumption" ⚠️

The opening of Part II ("everything listed here follows from the local rule with no additional assumption"), Chapter 5 ("the $\frac{1}{2}$ is the zero-point energy; it follows with no additional assumption"), and §6-2 ("not a single new assumption is needed") are all statements made **on top of the triple $(\mathcal{A}, \alpha, \omega)$ of Chapter 2**. And this book's $\mathcal{A}$ is the CAR algebra, including the complex structure $\Gamma^2 = -\mathbb{1}$ of Chapter 4. **That puts quantum structure in at the start.**

On this point the external document is right. The uncertainty relation (Chapter 5) and the Bell discussions (Chapters 20, 49) do not derive quantumness from a classical CA; they count **what follows automatically once a quantum algebra is adopted**. The book itself writes "superposition cannot be built ❌" in Chapter 18 and "what is tested is not determinism but commutativity" in Chapter 20, so there is no contradiction among the claims. No contradiction — but **the opening of Part II was too strong.**

The exclusion principle (Chapter 8) is different. The hard-core statistics of a binary cell is classical. That it is **fermionic** holds only in one dimension, where Jordan–Wigner is exact — which Chapter 8 already says.

Place	Tenth edition	Eleventh edition
Opening of Part II	follows from the local rule with no additional assumption	follows with no assumption beyond the triple of Chapter 2 (including the CAR algebra and complex structure)
Chapter 5	follows with no additional assumption	follows with no assumption beyond the $\mathcal{M}$ of Chapter 4
Preface § 序-1	causality, exclusion, the uncertainty relation ⋯ all emerge automatically	causality and hard-core statistics emerge from a classical CA; the uncertainty relation emerges automatically **once a quantum algebra is placed**

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Work deriving relativistic field equations from quantum CA (the D'Ariano–Perinotti line) also assumes quantumness from the outset, and the literature makes the distinction explicit 📚. This book makes the same distinction in its Preface.

85-5. Correction 5 — soften "cannot have a ground state" ⚠️

The box of §17-2, "the quasi-energies of a reversible CA densely cover the unit circle; there is no 'lowest'", is correct as a measurement (9701 points at $L = 8$, largest gap 0.0067). But, as the external document notes, the precise form of the problem is that "from the discrete update U alone, **a local, bounded-below energy generator H is not uniquely determined**." Infinitely many H satisfy $U = e^{-iH}$; whether any of them is local is a separate question.

§17-3, "Making the attribution precise", had in fact already written it this way ("the predicate 'spectral condition' does not exist in the language"; "not broken, merely not yet written"). What needs correction is the **force of the assertion** in the box of 17-2 and its consistency with 17-3.

	Tenth edition	Eleventh edition
Box of §17-2	there is no "lowest"	the generator that would define "lowest" is not uniquely fixed by U
Relation to Chapter 74	separate chapter	**another face of the same problem**. Chapter 74's "absence of a generator" concerns the generator of time evolution; Chapter 17 concerns the spectral condition of that same generator

> ⚠️ **Correction (thirteenth edition).** "§17-3 had in fact already written it this way" is wrong. §17-3 wrote that the inequality cannot be defined, which differs from the eleventh-edition form in the table. §87-7.

85-6. Correction 6 — it is the dispersion relation, not de Broglie's relation, that breaks ⚠️

The correction box placed at §6-8 in the structure proposal is recorded here as well. §6-5, "on the lattice de Broglie's relation is not exact", attributes the deviation wrongly. Defining the crystal momentum $p = \hbar k/a$, $\lambda = 2\pi/k = h/p$ is exact by definition; what breaks is $p = \gamma m u$, the relativistic relation among energy, momentum and velocity. §6-5's measured quantity $(\gamma\theta u - k)/k \simeq -k^2/6 - \theta^2/3$ is exactly that breaking. The conclusion "the deviation is (mass)²" is unchanged.

85-7. The shape that the six make together 💭

#	Target	Kind	What was excessive
1	Chapter 35	verdict changed	wrote G_3 as G — **dimension**
2	§36-2	**retraction**	read a necessary condition as sufficient — **coefficient** (the value of the canonical energy)
3	Chapter 37	null result inverted	read additivity as a wall — **coefficient** (the leading term of I/C_T)
4	Part II	wording restricted	did not count the algebraic assumption — **premise**
5	Chapter 17	verdict changed	"absent" should have been "not determined" — **uniqueness**
6	§6-5	verdict changed	attribution of the deviation — **definition**

Items 1–3 cluster in Part IX, and 2 and 3 share one shape. **The measured quantities are real and were measured correctly. What was excessive was reading them as fixing not only the "form" but the "coefficient".** The first law carries the form; the value of the canonical energy carries the coefficient. Additivity of mutual information carries the form; the leading term of I/C_T carries the coefficient.

> ⚠️ **Restriction (thirteenth edition).** "The first law carries the form" was also too strong. The first law is an identity and is no evidence for the form (§87-2). "The measured quantities are real" stands.

This is what Part XXV (Chapter 83) said from the side of theory: differences of vacuum energy are universal, absolutes are not; the form of the equation follows from symmetry, the coefficient from the microscopic rule. Chapter 60's "the area law appears as a scaling but not as a coefficient", Chapter 82's "we found where the generator lives, not its size", and items 2 and 3 of this chapter — **four times in the same place, the form appeared and the coefficient did not.** Added to Appendix D as D-8.

> The ✅marks of Part IX are ✅marks for the form.
> This book does not yet contain a single ✅for a coefficient.

85-8. One addition to the method 🆕

§82-6 recorded the three additions to the method made in the tenth edition (scouting, power calculation before running, checking the range of validity of the model family). This chapter adds one.

> **Before claiming that a quantity implies geometry, compute the same quantity in a model with no geometry.**

The counterexample of correction 2 is two qubits and fits on paper. Had it been written before §36-2, the equation "positive-definite quadratic form = second-order Einstein equation" could not have been set down. Correction 3 is the same: that the mutual information of independent copies is n-fold is known before running anything. **Ask "does this quantity also appear in a system without geometry?" before claiming geometry.** Added to the standard procedure from the fourth period on.

85-9. What this chapter does not claim ⚠️

- **No measured value of Part IX is retracted.** $c = 0.483$, the first-law ratio 1.000327, $S_{\mathrm{rel}} \propto \lambda^{1.9934}$, and the six-digit agreement of I/n all stand. What is retracted is the **inference** from them to "the second-order Einstein equation".
- **Holography is neither asserted nor denied.** Correction 3 moves ❌ back to ⚠️, not to ✅. The scaling of the leading term is unmeasured.
- **Not every point of the external documents is adopted as stated.** Correction 5 is confined to unifying the wording, since §17-3 already had it right. The phrase "general relativity was derived" does not occur in the text; Chapter 82 already wrote "response amplitude 12%, quantisation incomplete".
- **Apart from the recomputation of the counterexample (K400–K402), no new numerical experiment was run for this chapter.**

Part XXVIII The second law — does a quantum theorem fit on the lattice?

What this part sets out to clarify. Chapter 75 decomposed the arrow of a reversible CA into "atypical initial condition + coarse-graining + mixing", and said it had **not proved** the second law. Chiba–Yoneta–Hamazaki–Shimizu (2026) turned two forms of the second law into theorems for closed quantum many-body systems. Does their proof fit on a CA? If it does, what is specific to quantum mechanics?

Chapter 86 Does the second law fit on the lattice? ⚠️🆕

— Judging Planck's principle and the law of increasing entropy on reversible CA and QCA

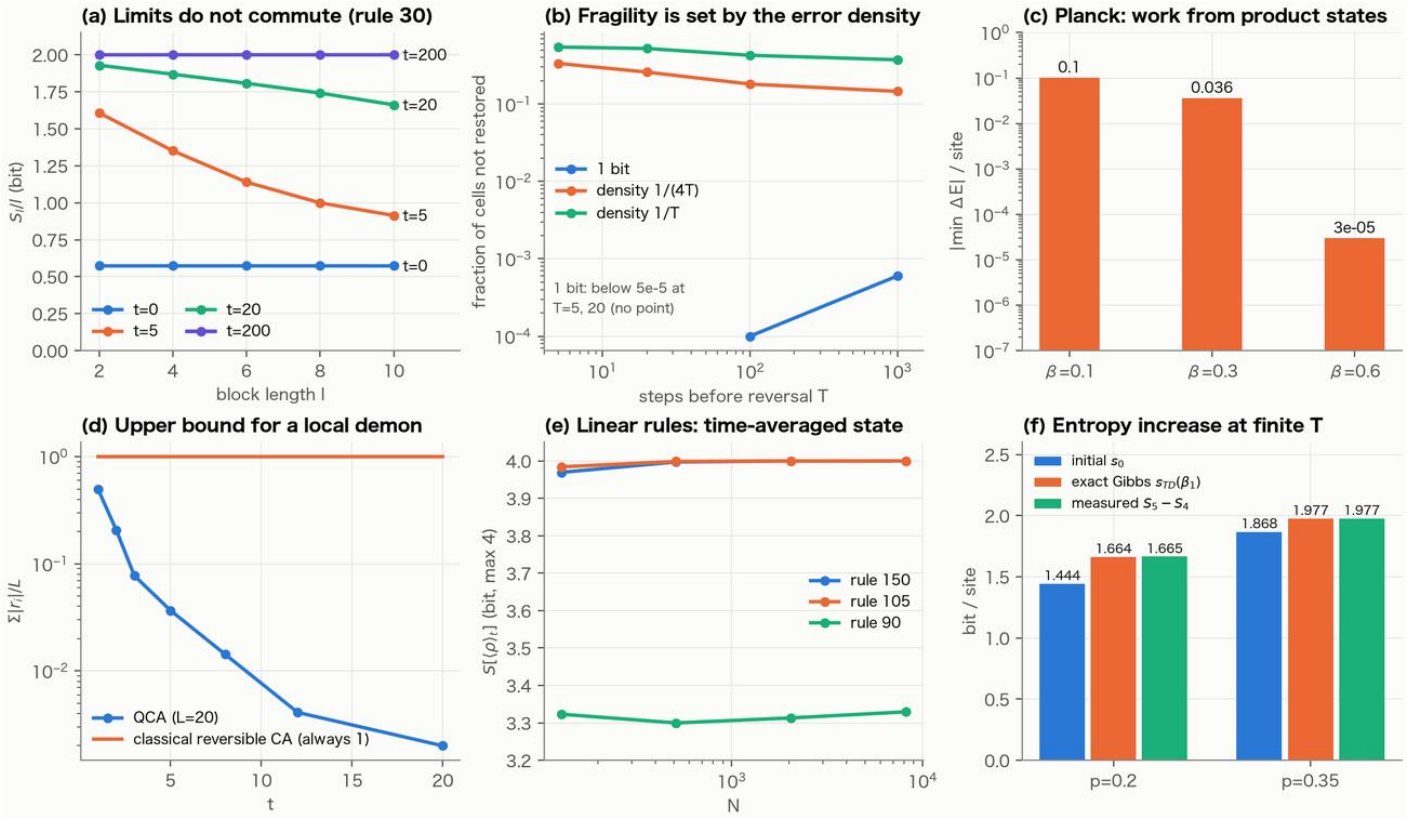


Figure 86-1. (a) K409. Rule 30, $s_0 = 0.573$. At fixed t, growing l heads to s_0; at fixed l, growing t heads to the maximum. (b) K416. Fraction of cells not restored after errors at the reversal. (c) K414. Work extractable by uniform protocols from product states of equal energy; exactly zero for the Gibbs state. (d) K419. Upper bound on what one layer of local operations can extract. (e) K418. Entropy of the time-averaged state from a Bernoulli start. (f) K415. Second-order Q2R type: initial entropy, exact Gibbs value, measured value.

The conclusion first.

(i) The second law for iMATE and macroscopic operations fits on reversible CA and QCA, in a stronger form because locality is exact.
(ii) But this form hardly uses noncommutativity. **It also holds on classical reversible CA.**
(iii) What is specific to quantum theory: local thermal equilibrium (MITE), passivity against a local demon, thermalization from a uniform pure state.
(iv) The existence and sign of energy, external operations, and thermalization itself do not come from the CA.
(v) The inequalities are time-symmetric. They do not explain the origin of the arrow.
(vi) The entropy law follows from a finite-size inequality. The classical–quantum difference appears in its boundary term (Araki–Lieb).

> ⚠️ **Restriction (thirteenth edition).** The "second law" of (i) is limited to Planck's inequality for iMATE and macroscopic operations, and the conservation and lower bound of entropy density. It does not include thermalization, the origin of the arrow, or a proof of the entropy law at finite temperature (§86-5, §86-14, §87-10).

86-1. What the paper proved 📚

The skeleton of the theorems of Chiba et al. is as follows (PRX Quantum **7**, 043023).

Ingredient	Content
Additive observable	$A_L = \sum_r \alpha_r$ (the locality l of α is independent of L)
Macroscopic equivalence	The densities of all additive observables agree as $L \to \infty$
iMATE	A state macroscopically equivalent to a Gibbs state. Pure states included
Macroscopic operation	Unitary generated by $H(t) = H - \sum_\mu f^\mu(t) B^\mu$, with B^μ additive and f^μ independent of L
Macroscopic entropy density	$s^{\mathrm{mac}} = \lim_l \lim_L S_{\mathrm{vN}}[\rho^{\mathrm{ave}}_{L\|l}]/l^d$ (ρ^{ave} is the spatial average of reduced density matrices)

There are four main results.

1. **Theorem 1:** Macroscopic operations with $t = O(L^0)$ preserve macroscopic equivalence.
2. **Corollary 3 (Planck's principle):** No extensive work can be extracted from an iMATE at $\beta \geq 0$ by a macroscopic operation with $t = O(L^0)$.
3. **Theorem 2:** $s^{\mathrm{mac}} = s^{\mathrm{TD}}$ on iMATE.
4. **Theorems 3 and 4 (entropy law):** Under a relaxation assumption, s^{mac} does not decrease.

The time $O(L^0)$ is optimal: there are counterexamples at $t = \omega(L^0)$ (Propositions 5, 6). The paper itself says they are close in spirit to Loschmidt's paradox.

Correspondence with this book's quantities. The $S(t)$ of Chapter 75 is the Shannon entropy of the empirical distribution of blocks of length 4. That is exactly the entropy of the diagonal of $\rho^{\mathrm{ave}}_{L|4}$. Chapter 75 was measuring $s^{\mathrm{mac}}_{l=4}$ without knowing it.

86-2. The embedding — one CA step is a macroscopic operation ✅🆕

A second-order CA $U : (a, b) \mapsto (b, f(b) \oplus a)$ can be written

$$U = \left[\prod_i \mathrm{SWAP}_{a_i b_i}\right] e^{-i\pi B}, \qquad B = \sum_i P^f_i(b)\, \frac{1 - X_{a_i}}{2}.$$

Both B and the generator of SWAP are additive, and f^μ is bounded and independent of L. So the form of the definition is met as it stands.

K411. On a ring $N = 3$ (6 qubits), for rules 30, 110, 150, the matrix error up to a global phase was at most 4×10^{-15}.

The "time group is $\mathbb{Z}$" of Chapter 17 is no obstacle here. A time-dependent generator realizing each step can be written. What cannot be written is a local, time-independent, bounded-below H fixed uniquely by U.

86-3. A response theorem without error ✅🆕

For a translation-invariant reversible CA or QCA α of radius r, the Heisenberg evolution of an additive observable of locality q is **exactly** an additive observable of locality $q + 2rt$. There is no Lieb–Robinson tail. Hence:

$$\boxed{\rho^{\mathrm{ave}}_{L|q+2rt} = \sigma^{\mathrm{ave}}_{L|q+2rt} \implies \mathrm{Tr}[\rho(t) A_L] = \mathrm{Tr}[\sigma(t) A_L] \quad (\text{equality at finite } L)}$$

Taking σ to be the Gibbs state on the ring, Planck's principle also becomes an exact finite-size inequality. Spatially non-uniform operations leave an $O(L^{d-1})$ correction at the boundaries.

K408. A 4-letter de Bruijn configuration ($L = 65536$, order 8) is a **single configuration** whose 8-block frequencies are exactly uniform: an exact version of the paper's Example 2. We compared integer frequencies at $q = 1, 2, 4$ for all 256 rules.

- Inside the threshold $t \le (8 - q)/2$ the mismatches were **0/2048**.
- Rules 30, 110, 45, 150, 90, 105 always broke one step past the threshold.

K417 (classical version of Proposition 5). Correlated pairs at distance $2T$ are brought together by a shift CA ($T = 64$, $L = 2^{20}$). S_l/l should fall once the pair distance $2(T - t)$ is shorter than the window l. At the measured times ($t = 0, 32, 62, 63, 64$), $l = 8$ fell at $t = 62$, $l = 4$ at $t = 63$, $l = 1$ at $t = 64$, as the light cone dictates. The final value is 1 bit/site, whereas Bell pairs reach 0. **In this example the size of the violation alone is larger for the quantum case.** §86-5 identifies where the difference comes from.

86-4. At finite time the entropy density does not move ✅🆕

We write the case where the inverse map also has radius r (second-order CA are such a case). The spatial average commutes with translation-invariant evolution. Classically the l-block at time t is a function of the $(l + 2rt)$-block at time 0, so

$$\boxed{S_{l-2rt}(0) \le S_l(t) \le S_{l+2rt}(0).}$$

On a QCA, $\alpha^t(\mathcal{A}_{C_l})$ is a factor of $\mathcal{A}_{C_{l+2R}}$ ($R = rt$), so Araki–Lieb gives

$$S_l(t) \le S_{l+2R}(0) + 2R \log D,$$

and likewise in reverse. Both corrections are boundary terms, so at fixed t

$$s^{\mathrm{mac}}[\alpha^t \omega] = s^{\mathrm{mac}}[\omega].$$

At finite time s^{mac} **neither rises nor falls**. This is also the entropy-side expression of the paper's Corollary 2 (no thermalization at finite time). Invariance under automorphisms commuting with translation, not only QCA, was obtained by Mohari along another route 📚. The value here is that it follows in a few lines from the CA light cone.

K409 (classical, $L = 2^{24}$, Bernoulli 0.05, $s_0 = 0.5728$). Violations of the inequality: **0/180**. S_l/l for rule 30:

t	$l = 2$	$l = 6$	$l = 10$	$S_{10} - S_9$
0	0.573	0.573	0.573	0.573
5	1.607	1.139	0.914	**0.571**
20	1.929	1.807	1.662	1.287
200	2.000	2.000	2.000	2.000

K410 (quantum, $p = 0.1$, $s_0 = 0.469$, 12 system + 12 ancilla qubits). For a non-Clifford brickwork at $t = 1$, $(S_{l+2} - S_l)/2$ for $l = 1, \dots, 5$ was 0.658, 0.634, $\mathbf{0.469}$, $\mathbf{0.469}$, $\mathbf{0.469}$. Violations of the upper bound: 0/7.

$$\boxed{\lim_{l\to\infty}\lim_{L\to\infty} S_l(t)/l = s_0 \;\neq\; \lim_{t\to\infty}\lim_{L\to\infty} S_l(t)/l = \log D \quad \text{(mixing rules)}}$$

The arrow is the failure of these two limits to commute. This is the precise form of Chapter 16's "the arrow lies in the partition". The $C(t)$ in $S_l(t) = s_0 l + C(t)$ is a boundary term and belongs with the "flux entropy" of Chapter 83.

86-5. A finite-size inequality and the entropy law ✅🆕

Setting. Let α be a translation-invariant reversible CA or QCA on a ring $\Lambda_L \subset \mathbb{Z}^d$, with inverse radius r'. For a protocol composed of finitely many uniform rules, add the radii. Write $\tilde{\rho}_L$ for the translation average of a state ρ_L. The l-block marginal of $\tilde{\rho}_L$ is $\rho^{\mathrm{ave}}_{L|l}$, and the translation average commutes with the evolution. $S_l(t)$ is the entropy of the l-block of $\tilde{\rho}_L(t)$.

$$\boxed{\begin{array}{l}\textbf{Theorem 86-1.}\quad \text{For } R' = r't,\ \ M = m + 2R',\ \ M = kl,\ \ M \le L: \\ \dfrac{S_l(t)}{l^d} \;\ge\; \dfrac{S_M(t)}{M^d} \;\ge\; \dfrac{S_m(0) - c\,(M^d - m^d)\log D}{M^d} \\ c = 0 \ \text{(classical)}, \qquad c = 1 \ \text{(quantum)}.\end{array}}$$

Proof. The left inequality is subadditivity of entropy. The right inequality:

- Classical: the m-block at time 0 is a function of the M-block at time t, so $S_m(0) \le S_M(t)$.
- Quantum: $F = \alpha^{-t}(\mathcal{A}_{C_m})$ is a unital matrix subalgebra of $\mathcal{A}_{C_M}$, so $\mathcal{A}_{C_M} = F \otimes F^c$, where F^c has matrix size $D^{M^d - m^d}$. In the Heisenberg picture $S_m(0) = S(\tilde{\rho}_L(t)|_F)$, and Araki–Lieb gives $S(F) \le S(FF^c) + S(F^c) \le S_M(t) + (M^d - m^d)\log D$. □

The entropy of the time-averaged state $\frac{1}{T}\sum_{t<T}\rho_L(t)$ is, by concavity, at least the average of the entropies at each time, and hence at least the average of the lower bounds at each time.

Corollary 86-2 (entropy law, $L \to \infty$ first). For a sequence representing a macroscopic state, at fixed l, t

$$\lim_{L\to\infty} \frac{S_l(t)}{l^d} \;\ge\; s_0 := \lim_{m\to\infty}\lim_{L\to\infty} \frac{S_m(0)}{m^d}.$$

Taking $k \to \infty$ removes the boundary term, since $R'/m \to 0$. **No condition on the rule or the initial state, and no thermalization assumption.** The same holds for time-averaged states and for any limit point as $T \to \infty$. Combined with the conservation of §86-4, s^{mac} is conserved at finite time and no finite-block density falls below s_0. Affinity and upper semicontinuity of the mean entropy (Lanford–Robinson 1968) 📚 lead to the same conclusion, but the proof did not need them.

Corollary 86-3 (bound on work). Let $H = \sum_r h_r$ (locality q) and $\beta > 0$. From the Gibbs variational principle on a block, with $e(t)$ the energy density ($d = 1$, entropy in nats),

$$e(t) \;\ge\; \frac{S_M(t) - \ln Z_M(\beta)}{\beta\,(M - q + 1)}.$$

Let e_0 be the initial energy density and $\beta > 0$ the corresponding inverse temperature. Using $e_0 = f(\beta) + s^{\mathrm{TD}}(e_0)/\beta$, the extractable work obeys

$$\boxed{\frac{W}{N} \;\le\; \frac{1}{\beta}\Big(s^{\mathrm{TD}}(e_0) - \frac{S_M(t)}{M}\Big) + \varepsilon_M, \qquad \varepsilon_M \to 0 \;\; (M \to \infty).}$$

With hidden correlations $s_0 < s^{\mathrm{TD}}(e_0)$, so this bound stays positive. **If $s_0 = s^{\mathrm{TD}}(e_0)$ (iMATE), then $L \to \infty$ at fixed t gives back Planck's principle.** In this limit alone, Planck's principle is a corollary of entropy conservation and the variational principle. The paper calls the two forms quantum-mechanically inequivalent, but its entropy law includes relaxation and rests on a different set of assumptions, so there is no contradiction.

Corollary 86-4 (time growing with L). If there is a scale $m_L \to \infty$ with $S_{m_L}(0) \ge (s_0 - o(1))\, m_L^d$ and $t_L = o(m_L)$, then Corollary 86-2 holds at $t = t_L$ too. The paper's counterexamples have $m_L \simeq 2T_L$, $t = T_L$, and do not satisfy $t_L = o(m_L)$.

This gives quantitative form, **within uniform rules**, to candidate (iii) of the paper's §XIV, "restrict to states with complex structures". The condition is that no hidden order lies inside the scale m_L, and it can be measured through S_m. For a pure state whose volume law $S_m \simeq s_0 m$ continues to $m \sim L/2$ (Page states, or energy eigenstates assuming ETH), the entropy law extends to $t_L = o(L)$. At $t \propto L$ only a lower bound loosened by a constant factor remains.

The difference among pure states. The empirical distribution of a single classical configuration is limited by $S_m \le \log L$, so this certificate only reaches $m \lesssim \log L / s_0$. A quantum pure state can reach $m \sim L/2$. But this is a **difference of certificates**. It does not mean a typical random configuration actually fails at $t \sim \log L$.

It saturates exactly — the quantum difference is the Araki–Lieb term. For hidden pairs (distance $2T$) set $m = 2T$, $t = T$, $M = 4T$.

	Lower bound on $S_M(T)/M$	Value reached
Classical ($c = 0$)	1 bit	**1 bit** (K417, K424)
Quantum, maximally entangled pairs ($c = 1$, $\log D = 2$ bit)	0	**0** (K425, the paper's Proposition 6)

Theorem 86-1 is reached with equality in both cases. The difference between the classical 1 bit and the Bell 0 bit of §86-3 is the second term of the inequality itself.

K423 (classical, $L = 2^{16}$). The empirical distribution of a single configuration is the state itself, so there is no statistical error. Rules 30, 110, 150, with Bernoulli and Loschmidt states: 4707 checks, **0** violations, smallest margin 1.2×10^{-2}.

K424. Hidden pairs at $T = 2$ ($L = 2^{20}$): for $m = 4$ the bound is 1.0000 and the measurement 1.0000; for $m = 6$ the bound is 0.9999 and the measurement 0.9999. $T \geq 3$ needs $M > 10$, beyond the capacity of a single configuration, so it was not measured. This is a live instance of the "difference of certificates" above.

K425 (quantum, $\mathbb{C}^4$ cells, $L = 10$ cells, translation invariant). Long-range pairs under the SWAP rule and generic gates: 16 checks, 0 violations. Under the SWAP rule every block has $S = 0$ at $t = d$ (and the pairs separate again at $t = d + 1$).

K426 (work, $p = 0.1$). For partially correlated pairs ($s^{\mathrm{TD}}(e_0) = 2H(p)$), the classical case measured $W = 0.1002$, bound 0.148. The quantum pair becomes pure in one cell ($S_1 < 10^{-14}$), and a local rotation gives $W = 2p = 0.2$, bound 0.296. **From the same correlation structure, quantum theory extracts twice the classical amount.**

K427 (translation average of a Haar state, $L = 7$ cells). Violations 0/4. But a positive lower bound appeared only at $t = 1$, $m = 3$ ($S_5/5 \geq 0.395$). At $L = 7$ the bound is negative at $t = 2$.

What it does not reach. The inequality needs $M \leq L$. It does not reach $t > (L - m)/(2r')$ in the paper's order ($T \to \infty$ at finite L). It does cover orbits of period at most $(L - m)/(2r')$, but in general **Assumption 2 of Theorem 4 does not go away.** The theorem is a lower bound only and says nothing about where above s_0 the state relaxes. Also, with $L \to \infty$ first, transport at $t \sim L$ is invisible, so the relaxation of macroscopically non-uniform states is not described. For non-uniform protocols the same argument should hold inside each macroscopic subsystem up to $O(L^{d-1})$ boundaries, but this chapter does not write it out.

86-6. Where is the energy? ⚠️🆕

What Planck's principle needs of H is **not that it generates the time evolution but that it be additively conserved**. Passivity of the Gibbs state holds for arbitrary unitaries, and permutations are among them.

K412. For all 256 second-order ECA we searched for additive conserved quantities up to locality q (excluding the trivial dimension 4^{q-1}).

	$q = 2$	$q = 3$	$q = 4$
Rules with a nontrivial conserved quantity	117/256	124/256	(named rules only)
Rules 30, 110, 45, 150, 105, 214	0	0	**0**
Rule 90	2	4	8
Rule 54	1	3	6

P244 "few" failed. This count does not separate statistical-mechanical energy from local conservation laws that obstruct statistical mechanics; Takesue (1989) addressed that distinction 📚. Many rules with conserved quantities look nearly inert, but this was not checked one by one.

For a rule with no conserved quantity the only equilibrium is $\beta = 0$. There Planck's principle becomes an equality (the paper's Corollary 4) and loses content. The saturation value 4.000 of Chapter 75 is this $\beta = 0$. It sits in the same place as the view that periodically driven quantum systems head to infinite temperature (Lazarides–Das–Moessner, D'Alessio–Rigol) 📚.

> Finite-temperature thermodynamics needs a conserved quantity. Element 3 of Chapter 40 (constraint) appears here too.

The sign. Q and $-Q$ are both conserved. $\beta \geq 0$ cannot be selected from inside the CA. The invariant statement is "one cannot move further from $\beta = 0$ on the same side". 💭 For the second law, the spectral condition of Chapter 17 matters at this one point, the sign. Continuous time itself was not needed for the theorems.

86-7. A finite-temperature model — second-order Q2R type ✅🆕

Take a rule of radius 2 (the second-order form of Q2R, widened to radius 2 in one dimension).

$$b_i' = a_i \oplus \big[h_i(b) = 0\big], \quad a_i' = b_i, \quad h_i = \sum_{d=\pm1,\pm2} \sigma^b_{i+d}, \qquad E = -\sum_i \sigma^a_i h_i(b)$$

Calibration. E was exactly conserved over 500 steps. For $q \leq 4$ the only nontrivial conserved quantity was E. The Gibbs marginals and s^{TD} were computed exactly by a transfer matrix (16×16).

K414 (Planck, noiseless). On the $L = \infty$ Gibbs marginals we evaluated ΔE exactly for 12568 protocols: 24 site permutations, 256 second-order rules, and their two-stage compositions.

Initial state	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.6$
Minimum ΔE, Gibbs	**0**	**0**	**0**
Minimum ΔE, product state of equal energy	-0.101	-0.036	-3×10^{-5}

The Gibbs state at $\beta = -0.3$, outside the theorem, gave -2.98. The minimum is not positive because the identity operation is included.

K415 (entropy law). Time average over $t = 2000$–8000 from a Bernoulli start ($L = 2^{18}$).

p	β_1	Initial s_0	Gibbs $s^{\mathrm{TD}}(\beta_1)$	Measured $S_5 - S_4$	L1 (vs Gibbs)	L1 (vs other start)
0.20	0.293	1.444	**1.664**	**1.665**	0.0043	0.0048
0.35	0.089	1.868	**1.977**	**1.977**	0.0056	0.0057

The "other start" is a single configuration (Euler ring) whose low-order block frequencies are exactly those of the product measure. The block length is 4 for $p = 0.2$ and 2 for $p = 0.35$, and the final-state comparison uses a different L. This is numerical support for the paper's Assumption 2 in one model, not a proof.

86-8. What only quantum theory gives ✅🆕

Classical configurations are in iMATE. The paper's Example 2 is a product state in the computational basis, that is, a bit string. The laws above need no entanglement. The difference shows up in two places.

(a) The local demon. The local state of a configuration is always pure (Bloch vector $|r_i| = 1$). **No part of a configuration is ever in local thermal equilibrium.** So MITE (thermal equilibrium locally too) does not exist on a classical CA, and the passivity of the paper's Proposition 15, "MITE + local control", has no classical counterpart.

K419. The most that one layer of local rotations can lower $\sum \sigma^z$ is $\sum |r_i|$.

t	1	3	5	12	20
QCA ($L = 20$)	0.497	0.078	0.037	0.0041	**0.0020**
Classical reversible CA	1	1	1	1	1

A classical demon that knows the configuration can extract everything. The paper's Example 5 is this calculation. 💭 In a closed universe the demon's memory is part of the system, so the classical second law is protected by Landauer's accounting (Chapter 15). For quantum MITE, even complete knowledge of the state extracts nothing by finite-time local operations. **What protects it is not the accounting but the state itself.** This is the reverse side of Chapter 18's "permutations create no entanglement".

(b) Thermalization from a uniform pure state. A translation-invariant configuration stays translation invariant forever under a translation-invariant rule (Chapter 1).

K420. Classically, the 4 uniform configurations became non-uniform within 100 steps under all 256 rules **0/1024** times. On a QCA, from $|0 \cdots 0\rangle$ ($L = 20$), S_l grows as follows.

t	$l = 2$	$l = 6$	$l = 10$
1	1.619	2.178	2.178
2	1.932	4.860	5.107
4	1.995	5.951	8.884
12	2.000	5.997	9.275

At fixed t it saturates in l ($\leq 4t$ bit, an area law), and with growing t it approaches l. The 9.275 at $l = 10$ is half of $L = 20$, hence a finite-size value. **The only source of entropy on a classical CA is the spatial disorder of the initial configuration.** Classically, this adds a floor to the "atypical initial condition" measured by K232 of Chapter 75.

86-9. Eigenstates — a CA version of ETH ⚠️🆕

The eigenstates of a permutation are Fourier sums over cycles, and their diagonal expectation values are **cycle averages**. The diagonal part of ETH thus becomes a combinatorial question: are additive observables equidistributed on each cycle? Because of degeneracies across cycles, this eigenbasis is not unique.

K421. State-weighted width of the cycle average of the bit density. The medians reproduce K234 of Chapter 75 (rule 30: 44, 147, 238, 588, 1024).

Rule	$N = 6$	$N = 8$	$N = 10$	$P(\lvert\bar{n} - \frac{1}{2}\rvert > 0.05)$ at $N = 10$
110	0.0291	0.0141	**0.0072**	0.0012
30	0.0542	0.0295	**0.0151**	0.0041
150 (linear)	0.0589	0.0510	0.0289	0.0759
90 (linear)	0.0833	0.0625	0.0500	0.1881

For every rule the largest deviation stayed at 0.5. **Weak ETH is supported; strong ETH is broken.** Short cycles of periodic configurations are the cause. Shiraishi (2018) analyzed this mechanism in a quantum emulation of a CA 📚, concluding that neither ETH nor large effective dimension explains the thermalization there. This section adds no new mechanism. In the paper's iMATE framework the relaxation assumption concerns macroscopic states rather than individual eigenstates, so this breaking is not an immediate contradiction.

86-10. Irreversible rules ✅🆕

The first half of Theorem 1 also holds on an irreversible CA (the pullback stays additive). Passivity and the entropy law use bijectivity and are lost.

K422 (Life, $L = 1024$, Bernoulli 1/2 = iMATE at $\beta = 0$).

t	0	1	20	500	2000
Density	0.499	0.274	0.164	0.056	0.037
$S_{3\times3}/9$	1.000	0.822	0.570	0.241	0.169

The entropy density keeps falling. This is the erasure accounting of Chapter 15 seen from the side of the state. **Reversibility is a premise of the second law.** Chapter 75 showed that reversibility alone is not enough; this section shows the law fails without it.

86-11. Corrections to Chapter 75 ⚠️🆕

(a) Restricting "a linear reversible CA is a counterexample to the second law". The paper's quantity is the **entropy of the time-averaged state**, not the time average of the entropy.

K418. Block length 4, maximum 4 bit, window $t = 2000$–4000.

Rule	Start	$N = 128$	512	2048	8192
150	Bernoulli 0.05	3.969	3.997	3.9998	4.000
105	Bernoulli 0.05	3.984	3.999	3.9999	4.000
90	Bernoulli 0.05	3.323	3.300	3.313	3.330

For rule 150 the fraction of time with $S < 3.9$ is $0.153,\ 0.048,\ 0.011$ ($N = 128, 512, 2048$), falling with N. The 0 at $N = 8192$ only means the recurrence is outside the window, so it is not counted. The L1 difference between final states from Bernoulli and Euler-ring starts was at most 0.0004 (0.0011 for rule 90).

With the localized seed start of Chapter 75, on the other hand, rule 150 falls as $3.42 \to 2.94$ ($N = 128 \to 512$). That state approaches the all-zero configuration as $N \to \infty$ and is not an iMATE at finite temperature.

> The recurrence of rule 150 is real. But starting from iMATE, the time-averaged state heads to maximal entropy.
> The counterexample of Chapter 75 concerns "a low-entropy localized seed". It is not a counterexample to Theorem 4.
> Rule 90 stops near 3.3 because it has conserved quantities, not for lack of equilibrium (whether it is generalized Gibbs is untested).

(b) An operational definition of "mixing". §75-16 left mixing undefined. A candidate is the paper's Assumption 2: "macroscopically equivalent states relax to the same macroscopic state".

(c) "One bit prevents the return" depends on system size. The observation in §75-11 concerns N smaller than the light cone.

K416 (rule 30, $L = 2^{20}$). The apparent $S_8/8$ of $z = U^{-T}x_0$ at $T = 1000$ is 1.9999 bit; the true value is 0.141 bit. Going forward T steps lowers Q from 0.500 to 0.020. The fraction of cells not restored:

T	1 bit	Density $1/(4T)$	Density $1/T$
5	0.0000	0.335	0.545
100	0.0001	0.181	0.426
1000	0.0006	0.146	0.373

P242 "one bit prevents recovery" failed at $T \ll L$. The fragility threshold is an error density $\sim 1/(vT)$. This attaches a CA number to candidates (i)(ii) of the paper's §XIV (exact preparation and operation are impossible). But the dependence on T is an observation at three points, not a measured law.

86-12. What does not come from the CA ⚠️💭

- **External operations.** A closed CA has no external $f(t)$. An internal controller reaches only inside its light cone, so uniform operations in L-independent time cannot be built from inside. The remaining candidate is a uniform clock layer embedded in the initial condition, but the energy exchange with that layer has not been written. Note too that it presupposes the "now" of Chapter 71.
- **Thermalization itself.** It remains an assumption. This chapter returned numbers for one model.
- **The origin of the arrow.** Planck's inequality holds for U and for U^{-1}. It imposes a condition on the initial state; it does not explain a low-entropy past.
- **Consistency with bundle I.** Additive observables presuppose lattice translation and equal-time surfaces. This second law lives on the **regular-lattice side** that this book gave up for Lorentz invariance. Moving it to the causal sets of Chapter 77 requires redefining additivity through statistical homogeneity, which has not been started.

86-13. Verdicts on the declarations ❌🆕

	Content	Verdict
P240	Exactly uniform inside the threshold, broken one step past	Correct (K408)
P241	Entropy inequality and convergence of the increment	Correct (K409)
P242	One-bit perturbation prevents recovery	**Failed** (recovers at $T \ll L$. K416)
P243	Time-averaged state of linear rules at least 3.5 bit	**Partly failed** (not reached with seed start and rule 90. K418)
P244	Few rules have conserved quantities	**Failed** (about half. K412)
P245	Demon: classical 1, QCA ≤ 0.05 at $L = 16$	Correct (K419)
P246	Q2R thermalization and entropy law	Correct (K415)

	Content	Verdict
P247	Sampled test of Planck	**Design failure** (no difference in the control, no power. K413. Rebuilt as K414)
P248	Width of cycle averages shrinks for mixing rules	Correct (K421). The surviving largest deviation was not declared
P249–P254	Life, QCA pure and mixed, rule 150, classical Proposition 5, $q = 4$	Correct (K422, K420, K410, K418, K417, K412)
P255–P258	Theorem 86-1 classical, saturation, quantum; bound on work	Correct (K423–K426)
P259	Positive lower bound at $t \geq 1$ for a Haar state	**Partly failed** (at $L = 7$ only $t = 1,\ m = 3$. K427)

K414 rebuilds P247 and was **run without a declaration**. Its results, control included, are as stated in the text, but it is the only major number in this chapter without a prior declaration.

In addition, the first run of K415 did not terminate because the Euler ring had 64 million edges. It was rerun with a shorter block length (**a design error**). K411 is the calibration of the addendum declaration, and it was correct.

Of 20 calibrations, 5 failed or partly failed (K412, K413, K416, K418, K427). Again the misses were on the side of dynamics, counting, and finite-size estimates, while the theorems held. It is the same asymmetry as Chapter 75.

86-14. What is not claimed ⚠️

- **The CA versions of the theorems have not been peer reviewed.** The proof of Theorem 86-1 is complete but limited to uniform rules.
- **Theorem 4 of the paper is not reproved without assumptions.** The order of limits differs (§86-5).
- **Corollary 86-4 is conditional.** The volume law is an assumption, and the loosened bound at $t \propto L$ has been seen numerically only up to $L = 7$.
- **Fermionic CA are not treated.** The quantum case of Theorem 86-1 uses the tensor-product decomposition.
- **The second law for QCA is not reproved for general Hamiltonian systems.** We confirmed that the frame of the proof fits and tested it numerically, nothing more.
- **The numbers are one-dimensional** (except Life in K422). Block length at most 10, one finite-temperature model, one QCA gate.
- **The search for conserved quantities stops at $q \leq 4$ and is limited to one-site translation.** Period-2 conserved quantities were not examined.
- **K414 is a computation on the $L = \infty$ Gibbs state.** The passage to single configurations relies on K408 and K415. The protocols are a small family, not all unitaries.
- **K419 is a bound for one layer of local operations.** The multilayer case of Proposition 15 relies on the paper's theorem.
- **Takesue (1989) was checked only as far as its abstract.**

86-15. Where we stand

> The question of Chapter 75, "where does the arrow live?", now has one theorem-shaped answer.
> The arrow lives in the noncommutativity of the limits $t \to \infty$, $l \to \infty$, which a finite-size inequality can measure.
> The form of the law does not care whether the algebra is commutative; the difference appears in the boundary term.
> **What quantum mechanics adds to the second law is robustness against a demon.**

Against the book's recurring theme, the obstacle here is not commutativity alone. The main form of the second law survives commutativity; what commutativity takes away is local thermal equilibrium. Regularity, for its part, ends up supporting the second law: additive observables and macroscopic operations both use translation. **How to reconcile the regularity given up in Chapter 77 with the regularity used here is the next question.**

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Part XXIX　Peer review of the twelfth edition — two retractions, and assertions cut down

> **What this part sets out to clarify.** An external referee report on the text of the twelfth edition has arrived. It checks the logic, formulas, coefficients and scope that can be judged from the text alone. It makes 26 points. Which are right, which are misreadings, and what changes in this book when they are fixed?

Chapter 87　Peer review of the twelfth edition ⚠️🆕

— A universal heading falls at the special value you did not try

The conclusion first.

(i) Of 26 points, 17 are valid, 9 partly valid, none wholly wrong.
(ii) Two retractions: "the form of Einstein's equations appears at linear order" and "$p \neq 2$ belongs to no finite level".
(iii) The top-priority number $I_5 = 0.487$ was misread and reproduces on rerun. But the claim of that section was wrong.
(iv) The checks also show that Chapter 15's value for Life is about $1/2.7$ of the loss from a uniform input.
(v) The checks K428–K435 of this chapter were run **without prior declaration**.

87-1. The report, and checking it

The report (`peer_review/第12版査読報告.md`) was written from the Markdown text alone, without the code. Each point was compared with the text, and whatever computation could settle was computed independently (`peer_review/pr_check.py`).

Check	Content	Main result	Section
K428	Interference order of the power family ($p \le 6$, $n \le 8$, exact with integer amplitudes)	$I_5 = 0$ at $p = 4$, $I_7 = 0$ at $p = 6$	87-3
K429	Non-negativity of the families of K221, K222	fractions with negative values 97.6%, 99.3%	87-3
K430	Exact formula for the dispersion of Chapter 6	closed form agrees to 8 digits at six points	87-5
K431	Uniform-input loss of Life and ECA (full enumeration)	Life 5×4: 0.2615 bit/cell	87-4
K432	Orbits of the second-order form and the first-order rule	agree on the first step only	87-4
K433	One-dimensional $\mathbb{Z}_2$ model without electric term	agrees with the free chain to 8.9×10^{-16}	87-6
K434	$H \ge 0$ for a reversible CA (64 states)	constructed, error 3.3×10^{-14}	87-7
K435	Cell counts of Chapter 35 in one geometry	surface/BH bound $= 4.000$	87-8

The original code of K223 was also rerun and reproduced the table in the text to the digits printed. **No other K number was rerun.**

87-2. Correction 1 — "the form of Einstein's equations appears at linear order" is retracted ❌🆕

§85-2 retracted "the second-order Einstein equation" with a two-qubit counterexample, and wrote that the first law is an identity of any family of states, so "that alone does not verify the Einstein equation". Yet the Preface and Chapter 87 (the closing at the time) kept "the form appears at linear order" and "satisfy Einstein's equations at linear order". **The same counterexample applies equally to the linear claim**, because the first law is the linear-order statement itself. The eleventh edition's correction stopped halfway.

The Faulkner–Van Raamsdonk derivation (§83-3) requires the first law for every region and orientation, and the RT dictionary. Chapter 36 measured one family of intervals, and constructed neither a bulk metric nor a linearized Einstein tensor. Moreover §58-3 measures that step (2), the locality of the modular Hamiltonian, fails on the free-fermion chain.

Chapter 36 measured **part of a necessary condition** for linear Einstein. Einstein's equations are not verified even at linear order.

No measured value is retracted. $c = 0.483$, the ratio 1.000327, the parabolic modular Hamiltonian and the locality of T_{00} stand. The verdict of Chapter 36 goes from ✅ to ⚠️, and the wording in the Preface and Chapters 38, 44, 83, 88 is changed. Verification needs what §85-2 wrote for second order: constructing the bulk metric and independently computing the residual of the tensor equation.

87-3. Correction 2 — "$p \neq 2$ belongs to no finite level" is retracted ❌🆕

(a) The report's misreading. The report called "$I_5 = 0.487$ at $p = 4$" impossible. In the table of the text, 0.487 is the value at $p = 3.0$. Rerunning the same code gives 0.4872, and there is no evidence of a fault in the evaluator.

(b) The claim was wrong all the same. $|\sum_i x_i A_i|^4$ is a polynomial of degree 4 in the indicator variables, so I_5 vanishes identically. That argument of the report is correct.

K428 (medians of $|I_n|$). The zeros at $p = 2, 4, 6$ are exact (rational arithmetic on Gaussian integers; maximum over 200 cases each: $|I_3| = 0$, $|I_5| = 0$, $|I_7| = 0$).

p	$\lvert I_3\rvert$	$\lvert I_4\rvert$	$\lvert I_5\rvert$	$\lvert I_6\rvert$	$\lvert I_7\rvert$	$\lvert I_8\rvert$
2	9×10^{-16}	2×10^{-15}	3×10^{-15}	5×10^{-15}	8×10^{-15}	2×10^{-14}
3	2.01	0.99	0.37	0.34	0.44	0.53
4	9.86	8.70	$\mathbf{7 \times 10^{-14}}$	6×10^{-14}	3×10^{-13}	4×10^{-13}
5	15.8	37.1	18.4	6.53	4.83	2.98
6	109	457	106	99.5	$\mathbf{6 \times 10^{-12}}$	7×10^{-12}

K223 tried only $p = 1,\ 1.5,\ 2.5,\ 3$. **P36 never tested its universal claim at an even p.**

(c) The correct form. Suppose I_{k+1} vanishes for all amplitudes. $\Delta_{h_1 \cdots h_{k+1}} f(0) = I_{k+1}$ with $f(z) = |z|^p$, $f(0) = 0$. From $\Delta_{\ldots, h+h'} f(0) = \Delta_{\ldots, h} f(0) + \Delta_{\ldots, h'} f(0) + \Delta_{\ldots, h, h'} f(0)$ the next order vanishes too, and from $\Delta_{\ldots} f(x) = \Delta_{\ldots, x} f(0) + \Delta_{\ldots} f(0)$ it vanishes at any base point. Being continuous, f is by Fréchet's theorem 📚 a polynomial of degree at most k. $|z|^p$ is a polynomial only when p is a positive even integer, so

$$\mu_p \text{ has a finite level} \iff p \in \{2, 4, 6, \ldots\}, \qquad \text{and then the level is } p$$

This derivation has not been peer reviewed. "Next to level 2 lies a region of no level at all" is wrong: levels 4, 6, ⋯ sit as points in the continuous family. That $p = 2$ is isolated among nearby p (§75-4) is unchanged.

(d) The level 3 "measure". The families of K221 and K222 do not guarantee non-negativity.

K429. Checking all subsets, 97.6% of the arbitrary Hermitian family and 99.3% of the arbitrary symmetric trilinear family take negative values (2000 cases each). They cannot be called measures. The conclusion can be rescued: $\mu = |\sum A|^2 \sum a$ $(a > 0)$ was non-negative in all 500 cases, with median $|I_3| = 1.32$ and $|I_4| \leq 7 \times 10^{-14}$. **"Level 2 is not a consequence of the axioms" survives for non-negative set functions.** Physical meaning, strong positivity and dynamics have not been constructed. "Genuine" and "nothing wrong with it" are withdrawn.

87-4. Correction 3 — the number and the reading in Chapter 15 ⚠️🆕

(a) The number. The text's $\log|X| - \log|F(X)|$ counts only the size of the image. It is merely a **lower bound** on the entropy loss $H(X) - H(F(X))$ from a uniform input (since $H(F(X)) \leq \log|F(X)|$).

K431. All image sizes of the text were reproduced. Values per cell:

System	Text	Loss from uniform input
Life 4×4	0.1171	**0.3070**
Life 5×4	0.0955	**0.2615**
ECA 110, $L = 18$	0.0764	0.1078
ECA 30, $L = 18$	0.0021	0.0028
ECA 90, 150, $L = 18$	0.1111	0.1111

They agree only for the linear rules because the preimages are of equal size. **"Life discards about 0.1 bit" underestimates the uniform-input value by about 2.7 times.** And that value concerns only the first step from a uniform input.

(b) The reading. The second-order form does more than change the definition of the state.

K432. Comparing first-order F and second-order orbits for rule 30 ($L = 9$), even with $x_{-1} = 0$ they agree only on the first step, and disagree on 19 of 20 steps. **The second-order form is a different reversible dynamics.** In this sense "reversibility is not a property of the rule but of the definition of the state" is too strong.

(c) Heat. What vanishes is logical erasure. Heat needs a bath, a temperature and a physical implementation, and the CA contains none of them. "Emits no heat" is to be read "has no logical erasure". 2.74×10^{-22} J is a conversion that assumes Landauer's bound, not a measurement. The same conversion of the uniform-input value gives 7.5×10^{-22} J.

87-5. Correction 4 — Chapter 6 can be written with exact formulas ⚠️🆕

As the report says, the boost of §6-4 does not come from a lattice symmetry. The exact dispersion $\cos\omega = \cos k \cos\theta$ is not the continuum hyperbola, and $(\omega, k) = (\theta\cosh\eta, \theta\sinh\eta)$ is an approximate agreement at small k, θ. §6-4 is to be read as **motivation**, not derivation. The measurements of §6-5 are lattice quantities and stand. And without any boost, everything has a closed form. From $u = d\omega/dk = \sin k \cos\theta / \sin\omega$,

$$\gamma = \frac{\sin\omega}{\sin\theta}, \qquad \gamma\theta u = \theta\cot\theta\,\sin k, \qquad \boxed{\frac{\gamma\theta u - k}{k} = \theta\cot\theta\,\frac{\sin k}{k} - 1}$$

K430. Agreement with numerics to 8 digits at six points (k, θ). This formula replaces the fit with $R^2 = 0.999993$. Two things follow.

1. $-k^2/6$ **has nothing to do with mass.** It survives at $\theta \to 0$. The box "deviation $\sim$ (mass)²" is right only for $k \ll \theta$. The "unchanged" of §6-8 is corrected too. The factor $\theta\cot\theta$ is the part that goes as the mass squared.
2. $v_\varphi v_g = 1 - \theta^2/3 - \cdots$ **takes** $k \to 0$ **first.** In general it is $(\omega/k)(\sin k\cos\theta/\sin\omega)$.

87-6. Correction 5 — constraints alone do not force interaction ⚠️🆕

The theorem of §12-3, "zero inter-site bilinears commute with the local constraint", is right for this model. But "impose a local conservation law and interactions are forced" goes too far.

K433. Take an open $\mathbb{Z}_2$ chain ($N = 4$) with no electric term, $H = \sum_b \tau^z_b (c^\dagger_b c_{b+1} + \text{h. c.})$, and impose the Gauss law $G_r = (-1)^{n_r} \tau^x_{r-1}\tau^x_r$. The physical sector is 8-dimensional and its eigenvalues agree with the even-parity free chain to 8.9×10^{-16}. In one dimension the gauge field can be removed. This model has a constraint and is free.

What carries the interaction is the constraint **together with the electric term**. The book's own formula $U = 2h[1 - \cos(2\pi/N)]$ gives $U = 0$ at $h \to 0$. The static flux sectors in higher dimensions (Kitaev 2006) 📚 are also free in each sector (not computed in this chapter). And a local conservation law in the sense of a continuity equation holds for free hopping too. What matters is a local constraint of Gauss type.

> A constraint is not sufficient for interaction. Terms not expressible as bilinears appear in constrained models with an electric term.

How the electric term entered the D_2 numbers of Chapter 12 was not checked in this chapter. "A CA explains more than reality does" is withdrawn.

87-7. Correction 6 — $H \geq 0$ can be written in discrete time too ⚠️🆕

K434. Building H by eigendecomposition from the U of second-order rule 30 (ring $L = 3$, 64 states) gives $\min \operatorname{spec} H = 0$ and $\|e^{-iH}e^{i\phi} - U\| = 3.3 \times 10^{-14}$. In a finite system this can be done for any unitary.

"One cannot define the inequality when trying to write $H \geq 0$" is wrong, and so is the statement of §85-5 that "§17-3 had already written it this way". The correct form is the one in the table of §85-5: **a local, bounded-below H is not uniquely fixed by U.** The locality of the H of K434 is not addressed.

87-8. Correction 7 — the cell counts of Chapter 35 ⚠️🆕

K435. The text compares the volume of a cube (L^3), one face (L^2), and the BH bound of a sphere of radius L (πL^2). $1/\pi$ is a product of that mixture. Take a sphere of radius L throughout.

	Cells	Ratio to BH bound
Bulk	9.92×10^{182}	8.25×10^{60}
BH bound $A/4$	1.20×10^{122}	1
Surface	4.81×10^{122}	**4.000**

That the gap of $\sim 10^{61}$ closes as a matter of orders of magnitude stands. "Saturates the bound" and $1/\pi$ are withdrawn. Since ℓ_P is used, this does not explain the weakness of gravity (already in §85-1). The subtitle is changed.

87-9. Assertions cut down

Place	Twelfth edition	Thirteenth edition
§1-2	"locality and homogeneity leave nothing but a CA"	continuity, product topology, finite alphabet and full shift stated
§1-5	"reversibilizes any rule", "no ground state"	a group on the state set must be chosen; it is a different dynamics; subject to Chapter 17's restriction
§2-2	"the Hilbert space does not factorize"	the local algebras are graded and the factorization is not local
Chapters 3, 9, 73	"kinematic general relativity completely reproduced", "reconstructs the metric"	local propagation speeds define an effective causal structure
Chapters 4, 5	vacuum, Dirac sea and uncertainty are one thing	limited to the geometry of Gaussian states; the sea needs H; sign of ρ_{vac} stated
Chapter 8	"exclusion ✅"	a limit on local occupation; anticommutation and antisymmetry are further input
Chapter 9	"nothing was put in by hand"	subject to the premises of §85-4
Chapter 20	"non-commutativity is what violates", "non-commutativity → Born rule → indeterminism"	premises listed; the Born rule is not derived
§24-1	sign of the anisotropy	definition stated
§24-3	"only one way out", "necessary"	a statement about exact invariance; effective invariance is another matter
Chapter 69	"This is the Hawking temperature"	the slope of a temperature for classical mode conversion; three values of κ, no error bars; intercept referred to §74-4
Chapters 70, 81	"the escape is closed", "957σ"	an order count inside assumptions such as single scalar, divergence form, analyticity; σ is formal, statistical errors only
Chapters 86, 88	"the second law was not a wall"	what fits is a limited form; thermalization does not fit

Place	Twelfth edition	Thirteenth edition
Chapters 46–56	section numbers one larger than the chapter	headings fixed; references in the text already matched the chapter numbers
Accompanying material	"everything is reproducible"	`code/` included in the release; the extent of rerunning stated

87-10. Points not adopted

- **A fault behind $I_5 = 0.487$ at $p = 4$.** A misreading; there is no evidence of a fault (87-3).
- **"$\oplus$ is not defined on an arbitrary finite alphabet".** Too strong: any finite set can carry $\mathbb{Z}_{|\Sigma|}$. It is enough to say a group must be chosen.
- **"Chapter 86 is not a proof of the second law at finite temperature".** Right for the entropy law. The finite-size Planck inequality (§86-3) is proved for states with the same marginals as Gibbs.
- **"Not reproducible from the text alone".** True of the text alone. The bundles exist. But leaving them out of the releases up to the twelfth edition was this book's fault.
- **The range of the section-number shift.** The report says Chapters 41–55; it is in fact Chapters 46–56.

87-11. One addition to the method 🆕

> **Before writing a universal heading, name the special values you have not tried.**

The universal $p \neq 2$ falls at even p; "constraints force" falls at zero electric term; "Life discards 0.1 bit" moved when the definition changed. All of this can be seen on paper before running anything. It is the same kind of habit as §85-8, "compute first in a model with no geometry".

87-12. What is not claimed ⚠️

- **K428–K435 had no prior declaration.** With no predictions, they are not in the tally of hits and misses.
- **The theorem of 87-3(c) and the formula of 87-5 are the author's derivations and have not been peer reviewed.**
- **Not every K number was rerun.** Of the 279 numbers from K100 up, 273 appear in the documents of the included bundles and K401–K406 in the eleventh-edition drafts (`code/v11/`). The script for K403–K407 is included; no script for the recalculation of K400–K402 could be found. But that only confirms they are recorded.
- **K433 is a one-dimensional model with $N = 4$.** It is not a recomputation of the model of Chapter 12 itself.
- **Numbers the report does not address individually, such as those of Chapters 13, 37, 77–80, were not checked.**
- **The report's criticisms of the statistics (correction for search, systematic errors) are not answered.** The σ of Chapter 80 and elsewhere remain uncorrected.

Chapter 88 Closing — rule and law

— how much of the universe can a simple rule write?

Conway's Game of Life showed that a simple rule can have complex behaviour.

This book shows **seven** things. Five by the ninth edition, a sixth added in the tenth, and a seventh in the twelfth.

First, keeping complexity in analysable form removes interaction. Local constraints with an electric term bring it back (§87-6), and quartic terms, the Mott gap, the exponential hierarchy and Cooper pairs follow in a chain. **Constraints make physics.**

Second, put the rule on the boundary and space itself stands up. The gap of 10^{61} in cell counts closes as a matter of orders, and necessary conditions for Einstein's equations can be measured (not a verification at either order; §85-2, §87-2).

Third, an "irreducible input" like the mass ratio, once decomposed, merges into a standard question of cosmology. Filtering gives a lower bound, generation gives the value, selection gives homogeneity, the horizon gives the timing. None answers "why this value" alone, but **the division of labour is now fixed**.

Fourth, every wall a CA failed to cross reduces to the same two properties.

Wall	Cause
Lorentz invariance (dim-4)	the lattice is **regular**
pinning of defects under expansion	same
Bell inequality	the algebra is **commutative**
absence of a ground state	time is discrete
sparseness	there are no constraints
the generator of gravity (Chapter 74)	**the advance operator is commutative**

> Discreteness itself was never the obstacle.
> The obstacles were **regularity** and **commutativity**.

Fifth, one of the three bundles now has a shape. Part XII took Bundle I — random discreteness, a causal set, collapse as physics — and returned a channel $\mathrm{diag}(a, a, a^2)$, a threshold $a_c = 2^{-1/4}$, an immunity theorem for photons and for spin zero, a rate $\gamma = C(k/m^2c^2)\tanh^2(\zeta/2)$ that is not a Lorentz invariant, and a named best experiment. What it did not return is a **number**: the scale k is still fitted rather than derived.

Sixth, give up regularity and the generator comes back. That is how far the tenth edition got. The deformation algebra of a regular CA is exactly commutative, and cannot enter the middle ground where GR lives — closure that depends on the metric (Chapter 74). But on discreteness with regularity given up — a Poisson-sprinkled causal set — **the deformation algebra closes**. The way it closes depends on q^{ab}, and both the diagonal and the off-diagonal components come back in the right form (Chapters 77-80).

> **The generator was not absent; it had gone invisible on a regular lattice.**

What comes back, though, is **only the form**. The amplitude is $R = 0.116 \pm 0.037$ — only 12% of the continuum. And it is a number that stands at the end of a hundred and forty-three calibrations, **after computing before running that "pinning w would take 5×10^2 core hours" and standing down**.

So the verdict takes the same shape as in the ninth edition, but is placed one step further along.

> the wish is granted in **form**, and the one number that would settle it is still missing.

That is a better position than it sounds. A theory that predicts a pattern with no free functions can be killed by a single pair of experiments — one photon, one massive particle, at the same proper time. A theory that predicts nothing cannot. And on this book's own side, the three-point fit $R \simeq 0.0105\,\ell T$ **dies within one section** if ℓ and T are varied independently.

Seventh, what fits on a CA of the second law is a limited form. Planck's inequality for macroscopic operations on iMATE, and the conservation and lower bound of entropy density, become finite-size inequalities on reversible CA (Chapter 86). They hold on commutative classical CA too. The arrow lives not in the rule but in the failure of the limits $t \to \infty$ and $l \to \infty$ to commute. What commutativity takes away is local thermal equilibrium, and **what quantum mechanics adds is robustness against a demon.** But the theorem is limited to uniform rules and has not been peer reviewed, and thermalization itself and the low-entropy past remain unexplained. Regularity here is a support. How to reconcile the regularity given up in Chapter 77 with the regularity used here is the next question.

And this book gets it wrong one time in two. Across the 217 calibrations from K198 to K399, 113 failed (72 wrong predictions, 32 errors of declaration or design, 9 bugs I introduced myself). This number is left in the body of the book rather than at the back because **the breakdown of the misses changed the method**. Being wrong about the world cannot be helped. Designing an experiment that cannot measure anything can be prevented before running it. The three things the tenth edition added to the method — reconnaissance, a power calculation before running, and checking the range over which the family of models is valid — all grew out of my own failures.

And this is the answer to the original wish.

> That the world runs on a simple law —— keep it.
> That the world is not deterministic —— keep that too.
>
> What must go is the assumption that ”simple law” and ”determinism” are the same thing.
>
> Give up regularity and Lorentz invariance follows, at the price of a stochastic law.
> Give up commutativity and Bell experiments agree, with indeterminism as the consequence.
> And give up regularity and **the generator of gravity comes back too**. The price is locality.
>
> **Simplicity is lost in none of them.**

The rule just keeps turning. **But which way it turns is not decided.**

Appendix A Table of principal numbers

Values from the first generation (forty-one experiments) and values newly computed for this book (🆕) are kept separate.

A-1. Geometry, symmetry, free fields

Item	Value
metric read-off	slope 1.00 ± 0.04, $R^2 = 0.999$
holonomy exponent	2.0002, alignment 1.000000
geometry does not generate interaction	$D_2 \sim 10^{-32}$
closure of the bilinear algebra	residual 10^{-15}
size of the island	$N = 3$: 0.238; $N = 12$: 10^{-6}
mass invariant	correlation 0.9997 at fixed m/v; localisation length to 0.01%
chirality	slopes ∓ 0.99179, sum 3.3×10^{-16}

A-2. Constraints and interaction

Item	Value
Gauss law forces quartic terms	residual 1.00 from the bilinear span (flat case 5.4×10^{-15})
dropping gauge invariance	$D_2:\ 3.07 \times 10^{-2} \to 9.08 \times 10^{-17}$
nonlinearity indicator	$D_2 \simeq 0.054\,\alpha$
on-site U	$U_q(N) = 2h[1 - \cos(2\pi q/N)]$
lattice Green function	$d = 1$ linear ($R^2 = 0.997$), $d = 2$ $\log r$ (0.9996), $d = 3$ $1/r$ (0.999)
dimensional transmutation	$\ln(1/\Delta)/(2\pi/U) \to 0.968$
hierarchy	$U \approx 0.11\,t$ gives 10^{-24}
Cooper pairs	$E_b < 0$; pair bandwidth $3.8\times$ single particle

A-3. Gravity and holography

Item	Value
Bekenstein bound	bulk exceeds by 1.97×10^{60}; boundary $0.318 = 1/\pi$
emergent Newton constant	$c = 0.483 \to G = 3.109$ (exact 3)
first law	ratio 1.000327
locality	1.0004 at $R = 13$; $R \sim \ln \ell$; decay length 2.05
second-order response (necessary condition only; retracted as "Einstein" in §85-2)	$S_{\mathrm{rel}} \propto \lambda^{1.9934}$, positive for all shapes
sparseness failure	$N(x < c/12) \sim e^{c/2}$; 2.6×10^{14} at $c = 64$

Item	Value
long-time evolution	purity exactly 1 even at $t = 500$

A-4. New calculations 🆕

Thermodynamics (Chapters 15–17)

Item	Value
Life image-size loss (uniform-input loss in A-8)	4×4: 0.1171; 5×4: 0.0955 bit/cell/step
ECA 90 / 150	exactly 2.0000 bit/step (algebraic)
second-order form	exactly 0
Landauer heat (300 K)	2.74×10^{-22} J/cell/step for Life
coarse vs fine entropy	$S_A : 0.27 \to 1.98$; S_{total} fixed at 4×10^{-12}
quasi-energies of a reversible CA	9701 points at $L = 8$; largest gap 0.0067
Floquet band wraps the circle	width 2π at $\theta = \pi/2$; gap 3.1×10^{-3}

Bell (Chapter 20)

Item	Value
Gaussian ceiling	$2.828427 = 2\sqrt{2}$ with two shared pairs
one shared pair	exactly 2.000000
restricted to commuting settings	exactly 2.000000 (exhaustive)
anticommuting triple, same state	2.828427 ($C = -\mathbb{1}_3$)
dichotomy defect (commuting case)	$\sin 2t$ (1.0 at $t = \pi/4$)
critical-chain ground state	max 1.993055
Ising ground state	1.680973 at criticality

Electromagnetism and relativity (Chapters 21–24)

Item	Value
lattice Bianchi identity	residual $\leq 3.6 \times 10^{-15}$
$\mathbb{Z}_N$ monopoles	36–53% of cubes for random configurations
$\mathbb{Z}_N$ total charge	0 mod N; multiples of N as an integer
lattice anisotropy coefficient	$(ka)^2/36$ (measured 2.777778×10^{-2})
anisotropy on a Planck lattice	$\sim 10^{-58}$ at 1 eV, $\sim 10^{-30}$ at 100 TeV
species-dependent limiting speeds	unchanged under blocking (9.900990×10^{-3} from 1 to 64)

Mass ratio (Chapters 27–33)

Item	Value
composite inertia	$M^*/(m_1+m_2) \to 1.2131$
saturation of binding and radius	$E_b \to 2.1683$, $R^*/a \to 1.4824$
range of the reduced mass	exactly a factor 2
dissociation of a moving atom	none (E_b grows with P)
Born–Oppenheimer threshold	$r \gtrsim 10$; no upper bound
transmutation sensitivity	$2\pi t/U = 57.12$ at $U/t = 0.11$
coupling spread needed for 1836	$\delta U/U = 13.16\%$
drive for domain competition	$d\varepsilon/d(m/v) \to -0.2500$

Domain walls (Chapter 29)

Item	Value
calibration: immobility at $h = 0$	3.4×10^{-13} bonds
calibration: energy conservation	2.1×10^{-16}
sweep in a closed reversible system	$199.5 \to 3.2$ bonds; no return
expansion at fixed cell count	stops after 12 bonds (PN pinning)
expansion adding cells	drift -195.6 (sweeps)
depinning threshold	$h_c = 1.868 \times 10^{-3}$ at width 6.25; independent of T
threshold scaling	$h_c \propto w^{-1.55}$
terminal velocity	$v = 0.30\, h/\nu$
resolution of selection	$\delta(m/v) \gtrsim 5 \times 10^{-3}$

Horizon (Chapter 30)

Item	Value
ratio in an infinite system	5.923
ratio at horizon $N = 12$	3.326 (compression 0.562)
condition for the ratio to stay within 4%	$N/\xi \gtrsim 20$
ratio at de Sitter temperature $T = 0.12$	11.698 (from 5.394 at $T = 0$)
unified condition	$H \lesssim \Delta$

Rigidity and protection (Chapters 31–33)

Item	Value
wall surface tension	$\sigma = 0.09319$ ($J = 1.5$)

Item	Value
surface-tension invariant	$\sigma/(\lvert V(\Delta_0)\rvert\, w) = 0.037 \pm 0.001$
coexistence bound	$\delta(m/v) < 0.48\%$
absence of rigidity (unconstrained)	$d\ln\Delta_0/dK \in [-3.92, -2.80]$
special ratio in the sector search	$r = 1$ only $(288 \to 295)$
exact flatness under constraint	residual 10^{-15} for $h \geq 1.8$; $E(0) = -hL$ exactly
threshold created by the constraint	$\Delta_c \simeq 0.79h - 0.94$
verification of $U_q(N)$	difference $\leq 4.4 \times 10^{-16}$ for all (N, q)
rigid invariant	$R(N;1,3)/R(N;1,2)$: 1.15883 at $N = 7$, $\mathbf{1.18517}$ as $N \to \infty$
t/h needed for 1836	1.0007 at $N = 15$, $q = 2, 6$
density of the ladder	112 rungs for $N \leq 15$; 4 within a factor 1.5 of 1836
overlap of orbital with the nucleus	only $2.47\times$ from $r = 1$ to 1024
protection by detuning	$P_{\max} = v^2/(v^2 + (\Delta/2)^2)$; saturates at $r \sim 30$
electron-capture condition 📚	$(m_n - m_p)/m_e = 2.5310$; $m_p/m_e > 725$

Part XII — collapse made physical 🆕

quantity	value
CSL window from XENONnT	$r_C \gtrsim 11.8$ nm (lattice origin excluded by 25 orders)
revival law of a hidden layer	$t^* = 1.032\, L_s + 2.617$, $R^2 = 0.99996$ (affine, not a power law)
CHSH from the quantum measure on a DAG	$2\sqrt{2}$ (saturates Tsirelson)
CHSH from the classical measure	2 exactly
sprinkling, optimised settings	$2\sqrt{2}$ in 746/746 realisations, deviation 1.3×10^{-15}
anisotropic erasure	$S_{\max} = 2\sqrt{1 + \eta_A^2 \eta_B^2}$ — no threshold
isotropic erasure threshold	$\gamma\tau_{\text{tot}} = \frac{1}{2}\ln 2 = 0.346574$
tilt of the protected axis (median)	0.000° / 44.17° / 59.21° for 1+1 / 2+1 / 3+1
Bloch map, 2+1	$\mathrm{diag}(0.712, 0.712, 1.000000000)$
Bloch map, 3+1	$\mathrm{diag}(0.690, 0.678, 0.656)$
collapse threshold (stable)	$a_c = 2^{-1/4} = 0.840896$; $-\ln a_c = (\ln 2)/4 = 0.1732868$
collapse threshold (unstable)	$\gamma\tau_0 < 2^{1/4} - 1 = 0.1892071$ (9.2% more forgiving)
longest chain	$L_{\max} \sim N^{1/(d+1)}$; measured $0.533/0.402/0.281$
photon protected component	1.000000000000000 exactly; $S_{\max} = 2\sqrt{1 + a^4} > 2$ always
rate	$\gamma = C\,(k/m^2c^2)\tanh^2(\zeta/2)$, $C = 4/3$ or 1
regime boundary	$\zeta_c = \operatorname{arcsinh}\sqrt{4D\tau}$ (depends on τ; no universal value)
free particle from rest	$-\ln a = (k/m^2c^2)^2\tau^2$; measured exponent 1.994, coefficient 0.979

quantity	value
frame dependence	lab 0.139407 vs co-moving 0.013441 on the same trajectory (factor 10.4)
pair CHSH	$S_{\max} = 2\sqrt{2}\, a_A a_B$ (verified to 8×10^{-6})
decay averaging	$\langle a \rangle = 1/(1+\gamma\tau_0)$, $\langle b \rangle = 1/(1+2\gamma\tau_0)$
best current bound	$D/\gamma_{\text{Planck}} < 2.47 \times 10^{-34}$ (Hensen 2015, orbital speed)
best proposed bound	$k < 4.71 \times 10^{-38}$ kg²m²s⁻³ (relativistic e^-e^-, γ_L=10, 1 km)

Part XIII — from field equations to dynamics 🆕

quantity	value
chirality sum rule (local $d(k)$, 8 seeds)	$\Sigma\chi = 0$ exactly; Chern-number return 3.1×10^{-15}
overlap operator, GW residual	1.9×10^{-15}; deviation from the GW circle 2.0×10^{-14}
overlap tail at $r = 4$	1.4×10^{-3} (Wilson is exactly 0 for $r \geq 2$)
central charge	$c = 1.000271$
modular long-range ratio $\|h(3)\|/\|h(1)\|$	regular lattice 0.331; causal set 0.406; irregular sampling 0.206
infrared suppression of non-locality	$\lambda^{-0.045}$ (effectively none)
first law on a causal set	six digits; all 224 modes active
relative entropy	positive; exponent 1.989; monotone
RT scaling	raw $S \propto N^{0.719}$; $p = -0.032$ at $\kappa = 10^{-2}$
sequential growth (classical)	volume law $S \propto N^{1.060}$; ordering fraction needs $p \propto N^{-0.84}$
complex percolation	$I_3 = 2.4 \times 10^{-16}$, $I_2 = 0.110$
growth axioms ($n \leq 5$)	covariance 6.6×10^{-16}, normalization 1.8×10^{-15}, Bell 4.6×10^{-16}
unique pure-phase point	$q = e^{\pm i\pi/3}$; $\text{amp}(C) = \exp[i(\pi/3)(L-I)]$; $\|\text{amp}\| = 1$ to all digits
maximum entropy	$8.481359 = \ln 4824$ (best on the classical line 8.117943)
correlation of $S = (\pi/3)(L-I)$ with S_{BD}	0.061–0.098 (with ordering fraction 0.963–0.979)
Bell causality of the BD action	identically 0 (0 of 88 isomorphism classes split)
normalization of BD growth	$\|\sum - 1\| \geq 1.98$ for every β (≥ 2.22 in 4D)
globally normalized quantum measure	$I_3 = 4.3 \times 10^{-16}$, $I_2 = 0.179$; microscopically uniform
Wick branch vs classical (indicator A)	0.1510 vs 0.1728 (N=6); 0.1481 vs 0.1906 (N=7)
Wick branch vs classical (indicator B)	0.1117 vs 0.0975; 0.1415 vs 0.1053 (**reversed**)
Chapter 6 supplement: dispersion	$\cos\omega = \cos k \cos\theta$ (spherical triangle)
localisation length	$\xi = 1/\text{arccosh}(\sec\theta)$ (matches numerics to all digits)
de Broglie deviation	$-k^2/6 - \theta^2/3$ ($R^2 = 0.999993$); $v_\varphi v_g = 1 - \theta^2/3 - \theta^4/45$

A-5. Principal numbers of the tenth edition (K198-K399) 🆕

Quantity	Value	Chapter
Non-commutativity of the advance operators on a regular CA	**exactly 0** (confluence)	74
Price of the trilemma: tail of the effective neighbourhood	grows as $\ln \rho$	76
Causal-set deformation algebra: detection of the off-diagonal part	9.3σ	80
Symmetry $\varepsilon_A(c) = \varepsilon_D(-c)$ (continuum)	5.6×10^{-16} (machine precision)	80
Shape agreement (over a factor 16 in μ)	$\chi^2/\text{d.o.f.} = 0.54$	80
Amplitude R = causal-set k / continuum k	$\mathbf{0.116 \pm 0.037}$	80
Empirical rule for R (a hypothesis)	$R \simeq 0.0105\,\ell T$, spread 13.6%	80
Density dependence of the floor	$\propto \rho^{-0.167}$	78, 80
Over-dispersion of the estimator	$\sigma_{\text{measured}}/\sigma_{\text{Poisson}} = 1.61$	80
Asymptotic exponent of a higher-order rule	$s = (D-2)/N$ with D odd	81
Gap in the reachable set	$(-1,\ 1/3)$, width 1.33	81
Vector monomials with even D ($D \le 9$)	**0** (against 30040 with odd D)	81
Escape route from dropping shift symmetry	$s = -1/(k+1)$; $k \ge 44$ gives a TF exponent of 90, ruled out at 957σ	81

A-6. Principal numbers of the eleventh edition (K400–K407) 🆕

Quantity	Value	Chapter
Two-qubit counterexample: $\Delta S/\Delta\langle K\rangle$ ($p = 0.3$; $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$)	$0.972074,\ 0.997192,\ 0.999719$ (analytic 1)	85
Two-qubit counterexample: D/ϵ^2	$2.366162,\ 2.379444,\ 2.380801$ (analytic 2.380952)	85
Discrete KG recurrence vs Dirac walk ($\theta = 0.05$–1.5)	$\le 1.3 \times 10^{-15}$	6-7
Reversibility over 600 steps forward and back	4.9×10^{-15}	6-7
$(\omega^2 - k^2 - \theta^2)/(k^2\theta^2)$ at $k = \theta = 0.05$	$-0.33344 \to -1/3$	6-7
Maximum group velocity	$= \cos\theta$ (six digits at five values of θ)	6-7
Wave-packet centroid velocity vs group velocity	relative difference 1.8×10^{-4}	6-7

A-7. Principal numbers of the twelfth edition (K408–K427) 🆕

Quantity	Value	Chapter
One CA step vs its macroscopic-operation form (ring $N = 3$; rules 30, 110, 150)	$\le 4 \times 10^{-15}$	86-2
Exact response agreement (inside the threshold, all 256 rules, de Bruijn configuration)	mismatches $0/2048$	86-3
Violations of $S_{l-2rt}(0) \le S_l(t) \le S_{l+2rt}(0)$ ($L = 2^{24}$)	$0/180$	86-4
Rule 30, $t = 5$: increment $S_{10} - S_9$ and s_0	0.571 and 0.5728	86-4

Quantity	Value	Chapter
Violations of Theorem 86-1 (classical, $L = 2^{16}$)	$0/4707$, smallest margin 1.2×10^{-2}	86-5
Saturation of Theorem 86-1 (classical, $T = 2$, $m = 4,\ 6$)	bound and measurement agree at $1.0000,\ 0.9999$	86-5
Violations of Theorem 86-1 (quantum)	$0/16$ (Haar, $L = 7$ cells: $0/4$)	86-5
Bound on work ($p = 0.1$)	classical $0.1002 \le 0.148$, quantum $0.2 \le 0.296$	86-5
Minimum ΔE, Gibbs state (12568 protocols; $\beta = 0.1, 0.3, 0.6$)	0	86-7
Minimum ΔE, product state of equal energy	$-0.101,\ -0.036,\ -3 \times 10^{-5}$	86-7
Entropy law, second-order Q2R type ($p = 0.20,\ 0.35$): initial $s_0 \to$ Gibbs, measured	$1.444 \to 1.664,\ 1.665$; $1.868 \to 1.977,\ 1.977$	86-7
Local demon bound $\sum \lvert r_i \rvert / L$ (QCA, $L = 20$, $t = 1, 20$; classical always 1)	$0.497,\ 0.0020$	86-8
Uniform classical configurations becoming non-uniform (all rules, 100 steps)	$0/1024$	86-8
Width of cycle averages ($N = 10$; rules 110, 30, 150, 90)	$0.0072,\ 0.0151,\ 0.0289,\ 0.0500$	86-9
Entropy of the time-averaged state (Bernoulli start, $N = 8192$; rules 150, 105, 90)	$4.000,\ 4.000,\ 3.330$ bit	86-11
Fraction of cells not restored (rule 30, $T = 1000$; 1 bit, density $1/(4T)$, $1/T$)	$0.0006,\ 0.146,\ 0.373$	86-11

A-8. Checks of the thirteenth edition (K428–K435, without prior declaration) 🆕

Quantity	Value	Chapter
$p = 4$: $\lvert I_4 \rvert$ (median), $\lvert I_5 \rvert$ (exact)	$8.70,\ 0$	87-3
$p = 6$: $\lvert I_6 \rvert$ (median), $\lvert I_7 \rvert$ (exact)	$99.5,\ 0$	87-3
Fraction with negative values (arbitrary Hermitian, arbitrary symmetric trilinear)	$97.6\%,\ 99.3\%$	87-3
Non-negative level 3 example: $\lvert I_3 \rvert$ (median), $\lvert I_4 \rvert$ (maximum)	$1.32,\ 7 \times 10^{-14}$	87-3
Exact formulas	$\gamma = \sin\omega / \sin\theta$, $\gamma\theta u = \theta \cot\theta \sin k$	87-5
Uniform-input loss (Life 4×4, 5×4, ECA 110, 30)	$0.3070,\ 0.2615,\ 0.1078,\ 0.0028$ bit/cell	87-4
$\mathbb{Z}_2$ chain without electric term ($N = 4$) vs free chain, eigenvalue difference	8.9×10^{-16}	87-6
$H \ge 0$ for a reversible CA: $\lVert e^{-iH} e^{i\phi} - U \rVert$	3.3×10^{-14}	87-7
Sphere of radius L: bulk/BH bound, surface/BH bound	8.25×10^{60}, 4.000	87-8

Appendix B Tools and conventions

B-1. Covariance formalism

Conventions: $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, $\langle \gamma_a \gamma_b \rangle = \delta_{ab} + i\Gamma_{ab}$, pure Gaussian $\iff \Gamma^2 = -\mathbb{1}$, $E = \frac{1}{4}\mathrm{Tr}(A\Gamma)$.

```
from scipy.linalg import schur
T, O = schur(A, output="real")
# fix the sign of every 2x2 block; forgetting this flips the energy
for k in range(0, n, 2):
    if T[k, k+1] < 0:
        O[:, [k, k+1]] = O[:, [k+1, k]]
```

B-2. Indicators

Indicator	Definition	Meaning
D_1	$\|\Gamma^2 + \mathbb{1}\|_F / 2N$	distance from the Gaussian manifold
D_2	RMS of connected 4-point functions	violation of Wick
α	quartic / quadratic energy	strength of nonlinearity

B-3. Tools added in this book 🆕

CHSH observables (parity-even, squaring to 1, mutually anticommuting):

$$\Sigma^A = (i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2), \qquad \Sigma_1\Sigma_2\Sigma_3 = -i\mathbb{1}$$

Correlators by Wick:

$$\langle (i\gamma_p\gamma_q)(i\gamma_r\gamma_s)\rangle = \Gamma_{pq}\Gamma_{rs} - \Gamma_{pr}\Gamma_{qs} + \Gamma_{ps}\Gamma_{qr}$$

Horodecki criterion: $\mathcal{S}_{\max} = 2\sqrt{\lambda_1 + \lambda_2}$ with λ_i the two largest eigenvalues of $C^{\mathsf{T}}C$.

Two-body problem on a lattice (centre of mass and relative coordinate do not decouple):

$$A_d = (t_1 + t_2)\cos\frac{P_d}{2}, \qquad B_d = (t_1 - t_2)\sin\frac{P_d}{2}$$

Projection to the physical sector: gauge-invariant Q preserves the physical space, so $P^\dagger[H, Q]P = [\hat{H}, \hat{Q}]$. Project once (failing to do so costs a factor 10^4 in runtime).

B-4. Numerical limits

Item	Limit
exact diagonalisation	2^{18} (sparse: 2^{20})
modular Hamiltonian (double precision)	$\ell \lesssim 10$
modular Hamiltonian (arbitrary precision)	$\ell = 40$ in 8.4 s (mpmath, 120 digits)
search for conserved quantities	weight $w \le 4$

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Item	Limit
full enumeration of state space 🆕	2^{20} (Life 5×4)

B-5. Useful identities

Avoiding cancellation (for a pure Gaussian state):

$$\mathbb{1} + \Gamma_A^2 = \Gamma_{AB}\Gamma_{AB}^{\mathsf{T}}$$

The right side is a product, not a difference, so $1 - \nu^2$ is obtained without loss of precision.

Correlations of the critical Majorana chain:

$$\Gamma_{ab} = \frac{2}{\pi(b-a)}\ (b-a \text{ odd}), \qquad 0\ (\text{even})$$

Mind the sign — writing $2/(\pi(a-b))$ flips h globally.

Appendix C Record of errors

The first generation made over thirty errors in forty-one experiments and **retracted a main conclusion four times**. This book retracted once more and failed calibration three times. The textbook keeps that posture.

C-1. Five retractions

Retraction	Content	Cause
14 → 15	"solvable models are non-Gaussian in physical space"	$[H, D_j] \neq 0$ and degeneracy
24 → 25	"$t/g \sim 10^{24}$ is fine tuning"	natural via dimensional transmutation
29 → 30	"on-site U vanishes to all orders"	the Gauss law had not been imposed
34 → 35	"the Einstein residual is physical"	wrong truncation of the local part
Chapter 29 🆕	**"a closed reversible system cannot move the wall"**	**the energy went into the wall's own kinetic energy**

C-2. Three calibration failures in this book 🆕

Where	Symptom	Cause
Chapter 29 v1	wall moves 5.15 bonds at $h = 0$; $\|\Delta E\|/\|E\| = 5.7 \times 10^{-2}$	initial dimerisation was not the true Δ_0
Chapter 29 v2	bulk values positive on both sides (kink dissolved)	**the Schur branch jumped at the zero mode**
Chapter 32 v1	$\|[H, G_j]\| = 4.0$	the "local charge" was not a Gauss law
Chapter 32 v1'	"plateau" in every setting	the minimum sat at the edge of the search range

C-3. Three verdicts revised 🆕

Item	First generation	This book	Reason
$U(1)$	❌ impossible in principle	⚠️ impossible microscopically, **emergence possible**	emergent photon in quantum spin ice 📚
Lorentz invariance	❌ tuning at 10^{-20}	❌ but **two distinct violations**	dim-6 harmless, dim-4 is the real one
Bell	not tested	❌ **commutativity fails; determinism is irrelevant**	the CHSH calculation

C-4. One null result whose meaning inverted 🆕

The sector search of §32-1 returned only $r = 1$ and was recorded, with disappointment, as "the degenerate point where no ratio exists". Chapter 34 inverted it — $r = 1$ **is the resonance, the one place where protection fails**.

> The search was correctly reporting not where the value is but **where the danger is.**

C-5. Recurring error patterns

Error	How it was caught
degeneracy of the ground space (seven times)	$\mathrm{defect}(\Gamma) \neq 0$, vanishing gap, zero modes
the tacit assumption that $R = 1$ is local	**we had the data and did not use it**
missing operators in the basis	coefficients came out independent of t
constraint not applied	the Gauss law is not optional
sign conventions	energy flipped relative to exact diagonalisation
minimum at the edge of the search range 🆕	interiority had not been checked
measuring outside the perturbative regime	non-monotonic response

C-6. The habit that helped most

> Prepare the falsifying inequality first.

Every error was caught not by "the result looks odd" but by **an output that was logically impossible**.

- Print variational bounds, purity, defects and $[H, Q]$ residuals every time
- Always calibrate against a limit with a known exact solution
- **Declare, before looking at the result, what would make you believe it**

Chapter 20 ran six checks, Chapter 29 seven, Chapter 32 seven — **all before seeing the result**.

> A search without calibration is meaningless.

C-7. The asymmetry of semi-decidability 🆕

Searches for conserved quantities and for rigidity are **semi-decidable in the weight**.

> A jump is decisive. No jump says nothing.

Chapter 32 **declared this before running**. Without the declaration one interprets a null result to taste afterwards.

C-8. Edition 3.0 — one retraction and eight calibration failures 🆕

Retraction. The bound $\gamma < 1.80 \times 10^{4}\,\mathrm{s}^{-1}$, $\gamma/\gamma_{\mathrm{Planck}} < 9.70 \times 10^{-40}$,
claimed from Hensen 2015, is withdrawn: it assumed $v \sim c$ while the NV spins are stationary.
Recomputed with the orbital speed it becomes $D/\gamma_{\mathrm{Planck}} < 2.47 \times 10^{-34}$ —
weaker by 5.4 orders. See §55-5.

Correction. The §32-5 invariant does not tend to 1.18517 but to exactly $32/27$; the
tabulated value is $I(N \simeq 40)$. See the correction box in §32-5.

Calibration failures (8 of 31 declared).

#	what failed	cause	mine or the model's
1	class-operator product order	`C@U` instead of `U@C`; K1–K6 all passed anyway	mine
2	"complete measurement diagonalises D"	records are required, not measurement	mine

#	what failed	cause	mine or the model's
3	revival scaling	affine relation fitted as a power law (4 attempts)	mine
4	sprinkling v1	ran on empty input, acceptance 0/24, still printed	mine
5	two-sided dephasing formula	$2\sqrt{1+\eta^2}$ instead of $2\sqrt{1+\eta_A^2\eta_B^2}$	mine
6	photon channel	the protected 1 was hard-coded — circular	mine
7	$\gamma = D\tanh^2(\zeta/2)$	axes cover the sphere, not the plane: $C = 4/3$	**the model's**
8	regime boundary ζ_c	rate-vs-rate instead of spread-vs-1; ζ_c depends on τ	**the model's**

Two new standing rules.

- A calibration suite that only checks invariants can pass on a wrong object. At least one calibration must check a quantity that is supposed to be **non-zero** (failure 1).
- A calculation that runs on empty input is wrong; every test against the empty set passes trivially. The next failure after "runs on empty" is "does not stop on empty." (failure 4).

A recurring index-convention error, now counted four times. `kron` puts subsystem 0 in the high bit while `(idx>>cell)&1` puts cell 0 in the low bit. This mismatch produced four separate bugs across this book. It is now checked by an assertion rather than by reading.

C-9. Failures since the ninth edition — 113 in total 🆕

Across the 217 calibrations from K198 to K399, **113 failed. A failure rate of 52%.**

Kind	Count	What it means
Wrong predictions	**72**	I was wrong about the world: a declared value, sign or magnitude missed
Errors in the declaration or the design	**32**	**I designed an experiment that could not measure the thing.** Preventable before running
Bugs of my own making	**9**	Code errors. Every one was caught by a calibration or by a value outside the reachable set

The painful column is the second. Four of those rewrote a chapter's conclusion once already.

	The error	How it surfaced
Ch. 79	Held λ fixed against A_0, so the contamination moved the exponent the wrong way	s_{eff} moved with A_0 in the direction opposite to the prediction
Ch. 79	Pushed A_0 to the spacelike ceiling and put the continuum inside its caustic	The target itself was 1.68, not the 2.33 I had claimed
Ch. 80	Read "width 2.93" as statistical error. It was **the model breaking** (the continuum was outside the family)	The ratio was below the $w \to 0$ limit
Ch. 80	Used "A_0 must be several layers thick" as a constraint for three sections without ever testing it	When measured, retention did not fall even at 1.1 layers

The error of designing an unmeasurable experiment can be prevented before running. That is what the three new habits of the tenth edition are for.

C-10. External verification of the tenth edition — one retraction, three changed verdicts, one inverted null result, two restrictions 🆕

After the tenth edition, two external documents (`Holographic_Gravity_From_Constraints_ja.md` and the Klein–Gordon study, both 2026-09-13) cross-checked Chapters 35–37 and 58–60, Chapters 6 and 17, and the opening of Part II. The text is in Chapter 85; here the entries are recorded under the categories of C-1 to C-5.

C-10-1. Retracted conclusion (added to C-1, the sixth)

Retraction	Content	Cause
§36-2	**"The second-order Einstein equation also holds"**	**A necessary condition was read as sufficient.** The first law and the positive quadratic onset of relative entropy are identities of any family of quantum states (Klein's inequality) and hold for independent qubit pairs (K400–K402). The bulk canonical energy had not been computed — as the limitations of step37 themselves said

C-10-2. Verdicts changed (added to C-3, three)

Item	Tenth edition	Eleventh edition	Reason
G of Chapter 35	$G = 3/(2c)$, no subscript	$G_3 = 3L/(2c)$	the metric is H^2 (a slice of AdS_3); Brown–Henneaux; $c \simeq 1/2$ is outside the large-c regime
Attribution in §6-5	de Broglie's relation breaks	$p = \gamma m u$ breaks; $\lambda = h/p$ is exact by the definition of crystal momentum	the measured $(\gamma\theta u - k)/k$ is a breaking of the dispersion relation
Assertion in §17-2	there is no "lowest"	a local, bounded-below H is not uniquely fixed by U	brought into line with §17-3 and Chapter 74

C-10-3. Null result whose meaning inverted (added to C-4, the second)

Chapter 37 recorded "raising c leaves I/n unchanged to six digits" as evidence of the sparseness wall. The mutual information of independent copies is additively n-fold, so this is **additivity, not a wall**. The holographic test is on the leading term of I/C_T when C_T is raised in a theory bound into one T_{ab} by interactions. **Unmeasured.** Verdict ❌ → ⚠️.

C-10-4. Wording restricted (two)

Place	What was too strong
Opening of Part II, Chapter 5, §6-2, Preface §序-1	"no additional assumption". The triple of Chapter 2 (CAR algebra, complex structure) is presupposed. The uncertainty relation follows automatically once a quantum algebra is placed; causality and hard-core statistics follow from a classical CA
Preface §序-1	"even the Einstein equation to second order emerges automatically". The form emerges at linear order; at second order only a necessary condition

C-10-5. Recurring error patterns (added to C-5)

Error	How it was caught
A quantity claimed to imply geometry was not first computed in a model without geometry 🆕	the two-qubit counterexample fits on paper
Additivity read as structure 🆕	that I/n agreed "exactly" was itself the evidence of independence
Dropped subscript (G for G_3)	checking the name of the metric showed it was a slice

C-10-6. The habit that helped most (added to C-6)

> Before claiming a quantity implies geometry, compute the same quantity in a model without geometry.

The counterexample of correction 2 could have been written before looking at the result; the additivity of correction 3 was known before running. This is the gravitational version of C-6's "prepare the checking inequality first".

C-10-7. Tally

	Count
Chapters cross-checked	opening of Part II; Chapters 5, 6, 17, 35, 36, 37, 58, 60, 82
Retractions	**1**
Verdicts changed	**3**
Null results inverted	**1**
Wording restricted	**2**
Numbers recomputed (K400–K402)	3
New experiments run for this book	0

Part IX was retracted once in the first generation ("the Einstein residual is physical", C-1, 34 → 35) and now a second time at §36-2. **The same Part, retracted twice.** Both have the shape "the measured quantity was right; the inference read from it was excessive." When the fourth period returns to Part IX, the habit of C-10-6 goes first.

C-11. Failures of the twelfth edition — four missed predictions, two design failures 🆕

This records the outcome of the declarations of Chapter 86 (P240–P259). The main text is §86-13. **There is no retraction.**

C-11-1. Missed predictions (the "dynamics and counting" side of C-5)

Prediction	Content	Outcome
P242 (K416)	A one-bit disturbance at the reversal prevents recovery	**Missed.** It recovers at $T \ll L$. The threshold is an error density $\sim 1/(vT)$
P243 (K418)	Time-averaged state of linear rules at least 3.5 bit	**Partly missed.** Not reached with a localized seed or by rule 90
P244 (K412)	Few rules have conserved quantities	**Missed.** About half of the 256 rules (117 at $q = 2$, 124 at $q = 3$)
P259 (K427)	Positive lower bound for a Haar state at $t \geq 1$	**Partly missed.** At $L = 7$ only $t = 1,\ m = 3$

C-11-2. Design failures

Item	Content	Response
P247 (K413)	The sampled test of Planck's principle had no power; the control showed no difference either	Rebuilt as the exact computation K414. **K414 was run without a declaration** (§86-13)
First run of K415	The Euler ring had 64 million edges and the run did not terminate	Rerun with a shorter block length

C-11-3. Errors fixed at the drafting stage (not counted)

Item	Error	Fix
Proof in §86-5	It stopped at a sketch via affinity and upper semicontinuity	Replaced by the finite-size Theorem 86-1, whose proof needs neither
Corollary 86-3	The work bound was written with s_0, which is wrong because hidden correlations give $s_0 < s^{\mathrm{TD}}(e_0)$	Rewritten with $s^{\mathrm{TD}}(e_0)$. It returns to Planck only for iMATE
Bound for time-averaged states	Written as "the same bound at the value $t = T - 1$"	Corrected to "at least the average of the bounds at each time"
Figure 86-1	Panel (b) plotted floor values as points; panel (c) drew bars with no substance	Removed the points and bars; panel (b) carries a note instead

C-11-4. Restrictions on Chapter 75 (correction boxes in the text)

§75-11 (one-bit fragility holds when the light cone covers the system), §75-13(a) and §75-14 (the linear-rule counterexample concerns a localized seed), §75-16 (a candidate definition of mixing). The measured values stand; what changed is how they are read.

C-11-5. Tally

	Count
Calibrations (K408–K427)	20
Failed or partly failed (K412, K413, K416, K418, K427)	**5**
of which missed predictions	4
of which design failures	1
Design error not counted as a calibration (first run of K415)	1
Retractions	0

Once more the misses were on the side of dynamics, counting and finite-size estimates, while the theorems held. It is the same asymmetry as Chapter 75.

C-12. Peer review of the twelfth edition — two retractions, one numerical correction, four changed verdicts 🆕

This records the external referee report (2026-11-06) on the text of the twelfth edition. The main text is Chapter 87.

C-12-1. Retracted conclusions (added to C-1, seventh and eighth)

Retraction	Content	Cause
Chapter 36 and elsewhere	**"The form of Einstein's equations appears at linear order"**	**The counterexample of §85-2 was applied only to second order.** The eleventh-edition correction stopped halfway
§75-7	**"$p \neq 2$ belongs to no finite level"**	**The universal claim was not tried at even p.** The pass of P36 is withdrawn

C-12-2. Number corrected

Item	Twelfth edition	Thirteenth edition	Reason
Life's bit-erasure rate	0.0955 bit/cell	Correct as a quantity of image size. The uniform-input loss is 0.2615	A lower bound was read as the loss

C-12-3. Changed verdicts (added to C-3, four)

Item	Twelfth edition	Thirteenth edition
Chapter 3	Kinematic general relativity completely reproduced	An effective causal structure
Chapter 12	Constraints force interaction ✅	A constraint with an electric term ⚠️
Chapter 35	The 10^{61} problem disappears ✅, $1/\pi$	The gap of orders closes ⚠️, ratio 4
§75-6	A genuine level 3 measure ✅	A family of set functions not necessarily non-negative ⚠️

C-12-4. Recurring error patterns (added to C-5)

Error	What caught it
A universal claim not tried at special values 🆕	For even p the polynomial can be written on paper
Applying a correction only where it was found 🆕	A counterexample that works at second order works at linear order too
Reading a lower bound as a value 🆕	Computing the uniform-input loss by full enumeration
Giving measured values strong names ("heat", "measure", "Hawking temperature")	Peer review

C-12-5. Errors on the side of the report

The top-priority point read $I_5 = 0.487$ as a value at $p = 4$. In the table of the text it is the value at $p = 3.0$. The range of the section-number shift is also Chapters 46–56. Yet beyond the misreading, the argument about $p = 4$ itself was correct and this book's heading was wrong.

C-12-6. Tally

	Count
Points	26
Valid	17
Partly valid	9
Wholly wrong	0
Retractions	**2**
Numerical corrections	**1**
Changed verdicts	**4**
Checks of this book (K428–K435, without prior declaration)	8

Appendix D Basso continuo — recurring themes

> **Basso continuo** is the Baroque term for the bass line that sounds without interruption from the first bar to the last.

D-1. "Choose the right basis and the structure appears" — but finding it is the hard part

Six occurrences: refermionisation, Clifford orbits, the monomial/polynomial classification, inside and outside the island, truncation of the local part, and projection to the physical sector.

D-2. Constraints appeared three times in different roles

Chapter	Role
12	**carry interaction** (a constraint with an electric term; §87-6)
36	**thin the states** (necessary for sparseness)
32	**quantise the coupling** ($U_q = 2h[1 - \cos(2\pi q/N)]$)

And all three times, what a constraint says **exactly** is only what is impossible (zero commuting bilinears; degeneracy only at $r = 1$; flatness only at $\Delta_0 = 0$).

D-3. Randomness appears in a fourth role 💭

$$\boxed{\text{Randomness buys Lorentz invariance.}}$$

Regular discreteness picks a frame; random discreteness does not. **And random discreteness naturally comes with a stochastic law.**

Chapter 12: constraints make interaction. Chapter 37: constraints make sparseness. Chapter 32: constraints quantise couplings. Part XI: randomness makes symmetry. **In each case, what looked like an extra turned out to be a requirement.**

D-4. Saturation at the same place, three times 🆕

Chapter	Quantity that saturates	Point
27	binding energy, radius, inertia	$r \sim 32$
33	protection by detuning	$r \sim 30$
27	reduced mass	factor 2 as $r \to \infty$

$$\boxed{\text{On a lattice with finite bandwidth, a mass ratio buys everything it can buy by } r \sim 30.}$$

D-5. Once a shock forms, the very notion of an exponent disappears 🆕

The caustic of Chapter 79, the condition $\max |\nabla f|_q < 1$ of Chapter 80, and the "real solution ceases to exist beyond a finite radius" of Chapter 81 — the same shape three times. **Outside the window of linear response, trying to measure an exponent is itself the mistake.**

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An exponent is defined only while the response has not saturated.

D-6. A constraint you have not tested must not be used as a constraint 🆕

From Chapter 79 onward I believed that "A_0 must be several layers thick" and let it pin the operating point. When Chapter 80 finally measured it, **retention did not fall even at 1.1 layers**. Three sections had been designed inside a constraint that was not there.

D-7. The reachable values are discrete, and the value you want sits in the gap 🆕

Chapter	The discrete set	The value that was wanted and absent
33	$I \geq 2$	$I = 1$
70	$s \in \{-1,\ 1/3,\ 3/5,\ 5/7, \ldots\}$	$s = 0$ (flat)
81	$s = (D-2)/N$ with D odd	the same; the gap is $(-1,\ 1/3)$

All three times a **dichotomy** grew out of it. **A degree of freedom that looks continuous turns out to be discrete** — the structure that recurs most often in this book.

D-8. Four times in the same place, the form appeared and the coefficient did not 🆕

Chapter 60 (the area law appears as a scaling but not as a coefficient), Chapter 82 (where the generator lives, but not its size), Chapter 83 (the form of the equation follows from symmetry; the coefficient is non-universal), Chapter 85 (the first law carries the form; the canonical energy carries the coefficient). **This book's ✅ marks are ✅ marks for form; it does not yet contain a single ✅ for a coefficient.** These are the four faces, in this book, of what the Casimir effect said from the start: differences are universal, absolutes are not.

> ⚠️ **Restriction (thirteenth edition).** "The first law carries the form" went too far. The first law is an identity that holds in systems with no geometry, and is no evidence even for the form (§87-2).

References

> ⚠️ Bibliographic details are from memory and need checking. 📚 marks items not verified here.

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RG and hierarchy: Shankar (1994); Cooper (1956); Lieb & Wu (1968)

Lattice gauge theory and emergent gauge fields: Kogut & Susskind (1975); Fradkin & Shenker (1979); Kitaev (2003); Hermele, Fisher & Balents (2004); Levin & Wen (2005)

Entanglement and gravity: Ryu & Takayanagi (2006); Casini, Huerta & Myers (2011); Lashkari, McDermott & Van Raamsdonk (2014); Faulkner et al. (2014); Swingle (2012); Almheiri, Dong & Harlow (2015)

Modular Hamiltonians: Bisognano & Wichmann (1976); Peschel (2003); Eisler & Peschel (2017)

Holography: Bekenstein (1973); 't Hooft (1993); Susskind (1995); Maldacena (1998)

Thermodynamics and information: Landauer (1961); Bennett (1982); Bérut et al. (2012)

Bell's theorem: Bell (1964); CHSH (1969); Cirel'son (1980); Fine (1982); Horodecki³ (1995); Hensen et al. (2015); Giustina et al. (2015); Shalm et al. (2015)

Naturalness of Lorentz violation: Collins, Perez, Sudarsky, Urrutia & Vucetich (2004); Kostelecký & Russell (2011)

Causal sets: Bombelli, Lee, Meyer & Sorkin (1987); Rideout & Sorkin (1999); Sorkin (1991, 2007); Surya (2019)

Objective collapse: Ghirardi, Rimini & Weber (1986); Pearle (1989); Diósi (1989); Penrose (1996); Bassi et al. (2013); Tumulka (2006)

Informational reconstructions: Hardy (2001); Chiribella, D'Ariano & Perinotti (2011); Masanes & Müller (2011)

Everett interpretation: Everett (1957); Deutsch (1999); Wallace (2012); Sebens & Carroll (2018)

Wolfram physics: Wolfram (2002, 2020); Gorard (2020)

Domain walls and cosmology: Zel'dovich, Kobzarev & Okun (1974) 📚; Starobinsky & Yokoyama (1994) 📚

Universality of rule 110: Cook (2004) 📚

The interference hierarchy and difference equations: Fréchet (1909) 📚

The second law and statistical mechanics of CA: Chiba, Yoneta, Hamazaki & Shimizu (2026); Lanford & Robinson (1968) 📚; Mohari (arXiv:1705.11038, math/0701186) 📚; Shiraishi (2018) 📚; Takesue (1989) 📚; Kari & Taati (2015) 📚; Lazarides, Das & Moessner (2014) 📚; D'Alessio & Rigol (2014) 📚

Revision history

- **Note.** Chapter numbers in these entries are **the numbers in force at that edition**. Because the fifth edition inserts a new Chapter 33, everything from the old Chapter 33 onwards has shifted by one (old Chapter 68, "Closing", is now Chapter 88).
- **13.0** (2026-11-06) Part XXIX "Peer review of the twelfth edition — two retractions, and assertions cut down" (Chapter 87), Appendices A-8, C-12 added. An external referee report was checked: **two retractions (linear-order Einstein, §75-7), one numerical correction (Chapter 15), four changed verdicts. K428–K435 (without prior declaration).** Correction boxes in Chapters 1–6, 8, 9, 12, 13, 15, 17, 20, 24, 35, 36, 69, 75, 81, 85, 86. Section numbers of Chapters 46–56 fixed. `code/` included in the release. **Closing is Chapter 88.**
- **12.0** (2026-11-06) Part XXVIII "The second law — does a quantum theorem fit on the lattice?" (Chapter 86), Appendices A-7, C-11 added. **K408–K427, P240–P259; 20 calibrations, 5 failed or partly failed, no retraction.** Theorem 86-1 (the author's derivation, not peer reviewed, uniform rules only). Addition or restriction boxes in Chapters 14, 40, 75, 83. A seventh item in the closing. **Closing is Chapter 87.**
- **11.0** (2026-09-13) Chapter 6 addendum (§6-7 discrete Klein–Gordon equation, §6-8 correction box), Part XXV "Differences and absolutes" (Chapter 83), Part XXVI "Why is gravity weak?" (Chapter 84), Part XXVII "External verification of the tenth edition" (Chapter 85), Appendices A-6, C-10, D-8 added. **K400–K407. One retraction (§36-2), three changed verdicts, one inverted null result, two restrictions of wording.** Correction boxes in Preface-1, the opening of Part II, Chapters 5, 6-5, 17-2, 35, 36-2, 37. The Preface-2 reading order, which had stopped at Part XII, is brought up to date. **Closing is Chapter 86.**
- **10.0** (2026-08-24) Parts XIX-XXIV (Chapters 74-83) added. **K198-K399: 217 calibrations, 113 failures.** Part XIX (Chapter 74): the down-set advance operators of a CA are **exactly commuting**, so the algebra cannot occupy the middle ground where general relativity lives — the place where the bracket closes on the metric. The intercept of K160 is not explained by a greybody factor. Part XX (Chapter 75): the hierarchy of measures and an exact Loschmidt test. Part XXI (Chapter 76): **discrete, strictly local and Lorentz invariant cannot all three hold**; dropping regularity brings Lorentz back but breaks locality logarithmically. Part XXII (Chapters 77-80, the largest single question in the book): **on an irregular causal set the deformation algebra does close.** It closes on q^{ab}, and even the off-diagonal component comes back in the right shape ($\chi^2/\mathrm{d.o.f.} = 0.54$). What does not come back is the amplitude: $R = 0.116 \pm 0.037$. Part XXIII (Chapter 81): the escape route left open in Chapter 70 (nonlinear and higher-order rules) is closed. **What forbids a flat curve is parity** — $\mathbf{F}$ is a vector, so the total number of derivatives D is odd, and flat would require $D = 2$. The dichotomy strengthens into one with a gap $(-1, 1/3)$. Part XXIV (Chapter 82): the verdict on the three gaps, and the three habits added to the method (reconnaissance, a power calculation before running, and checking that the model family is valid). The closing moves to Chapter 83. Twenty-two new figures (the twenty of the new chapters plus two explanatory figures for Chapters 1 and 2), in Japanese and English.
- **9.0** (2026-08-19) Part XVII, "Entanglement — where does the correlation come from?" (Chapter 72), added. In the stabilizer formalism of a Clifford CA (integer arithmetic): a shared past is necessary for correlation (0 exceptions); the rate is set by the boundary and not the volume (1.0 against 0.5 bits/layer); and any local operation here leaves the distant reduced state completely unchanged (200/200). Still CHSH reaches $2\sqrt{2}$ with no-signalling at 5.6×10^{-17}; $I_3 = 0$, $I_2 \neq 0$. The "distant state" depends on the foliation (trace distance 0.500) while the correlations do not — collapse is bookkeeping. Calibrations K185-K197; two failures (a wrong declaration and the choice of gate). The balance sheet moves to Part XVIII / Chapter 73 and the closing to Chapter 74. One new figure.
- **8.0** (2026-08-18) Part XVI, "What is simultaneity" (Chapter 71), added: what a CA's synchronous update means causally. Refoliation invariance holds across 2720 trials with 0 disagreements; it rests only on the rule seeing its parents, and breaks 300/300 once the rule reaches past the light cone. On a ± 1 light cone the only antichains covering every cell are flat (verified exhaustively), while on a sprinkling the "now" can be tilted — what pins it is regularity, not discreteness. The "now" is a thick layer (median 3 steps). Calibrations K173-K184; one failure, a bug in my own test harness. The balance sheet moves to Part XVII / Chapter 72 and the closing to Chapter 73. One new figure.
- **7.0** (2026-08-18) Part XV, "Galaxy rotation curves — must dark matter be assumed?" (Chapter 70), added: if the local flux is analytic, the asymptotic slope of v^2 can only be one of $\{-1, 1/3, 3/5, \ldots\}$, and flat lies in the gap. Flatness requires $m = 1/2$ uniquely, which is non-analytic. The lattice Green's function correction is $\frac{5}{8} K_4 / r^2$ with zero angular average. Baryonic Tully-Fisher exponent 4 (observed 3.85 ± 0.09, 1.67σ). Calibrations K162-K172; three failures, one of them a sign bug. The balance sheet moves to Part XVI / Chapter 71 and the closing to Chapter 72. One new figure.
- **6.0** (2026-08-15) Part XIV, "Horizons — an acoustic black hole and Hawking radiation" (Chapter 69), added: a flow put into a position-dependent local rule makes an acoustic horizon, and the radiation off it is measured to be thermal, at temperature $\kappa/2\pi$, with $|\beta|^2$ equal to the Planck distribution itself. The lattice has no trans-Planckian problem. Calibrations K149-K161; four failures, three of them predictions of mine being wrong. The balance sheet moves to Part XV / Chapter 70 and the closing to Chapter 71. One new figure and one animation (in both languages).

- **5.0** (2026-08-14) Chapter 33, "A hidden dimension — tetrahedra, diamond, superlattice", added to Part VIII: the mechanism whose conclusion §32-5 had been quoting is now developed in the main text. Part XIV, "What the real universe looks like from here" (Chapter 69), added, taking stock of Lorentz invariance, Bell, causality and mass. The closing chapter is now Chapter 70. Two new figures (in both languages).
- **4.0** (2026-08-14) Part XIII added (Chapters 57-67): derivability of Einstein's equations and the Standard Model; the modular Hamiltonian, first law and Ryu-Takayanagi on a causal set; classical and quantum sequential growth; the axiom test; the reverse test of the Benincasa-Dowker action; global normalization; Wick rotation. A supplement to Chapter 6 (6-2 to 6-6, de Broglie waves). The closing chapter moved to Chapter 68. **A descriptive subtitle was added to all 69 chapters.** Eleven new figures (in both languages).
- **3.0** (2026-08-13) Part XII added (Chapters 45-56): collapse made physical, from the causal set to a proposed experiment. The closing chapter moved to the end as Chapter 57. §32-5 corrected ($32/27$, not 1.18517). One retraction (§55-5) and eight calibration failures recorded in Appendix C-8.

Edition	Date	Content
—	2026-08-06/07	First generation, The Universe Drawn by the Game of Life, eds. 1.0–6.0 (forty-one experiments)
1.0	2026-08-09	Monograph, first edition; reorganised systematically; thermodynamics, Bell, electromagnetism and Lorentz newly computed
2.0	2026-08-10	Chapter 0 added (six figures). Fully reordered for study. Algebraic formulation (Chapter 2), corrected attribution for Bell (Chapter 20), and the full mass-ratio development (Part VIII: filtering, generation, selection, horizon, rigidity, protection). 57 figures placed.
2.0-en	**2026-08-10**	**English edition. Chapter 0 figures and the five conceptual diagrams regenerated in English (18 figures rebuilt).**
3.0	2026-08-13	Part XII (Chapters 45-56); closing chapter moved to the end; Chapter 0 added
4.0	2026-08-14	Part XIII (Chapters 57-67) and the Chapter 6 supplement; subtitles on every chapter; 78 figures
5.0	2026-08-14	Chapter 33 (hidden dimension) and Part XIV (balance sheet); closing is Chapter 70; 80 figures
6.0	2026-08-15	Part XIV (acoustic horizon and Hawking radiation, Chapter 69); closing is Chapter 71; 81 figures + 1 animation
7.0	2026-08-18	Part XV (galaxy rotation curves, Chapter 70); closing is Chapter 72; 82 figures + 1 animation
8.0	2026-08-18	Part XVI (what is simultaneity, Chapter 71); closing is Chapter 73; 83 figures + 1 animation
9.0	2026-08-19	Part XVII (entanglement, Chapter 72); closing is Chapter 74; 84 figures + 1 animation
10.0	2026-08-24	Parts XIX-XXIV (Chapters 74-83); K198-K399; closing is Chapter 83; 108 figures + 1 animation
11.0	2026-09-13	Chapter 6 addendum; Parts XXV–XXVII (Chapters 83–85); Appendices A-6, C-10, D-8; K400–K407; one retraction; closing is Chapter 86; 108 figures + 1 animation
12.0	2026-11-06	Part XXVIII (the second law, Chapter 86); Appendices A-7, C-11; K408–K427; no retraction; closing is Chapter 87; `figures/` holds 109 files (107 PNG plus the animation as gif and mp4)
13.0	**2026-11-06**	**Part XXIX (peer review of the twelfth edition, Chapter 87); Appendices A-8, C-12; K428–K435; two retractions; closing is Chapter 88; `figures/` holds 109 files; `code/` included**

Accompanying material

File	Content
`code/chain.py`, `j_transport.py`, `manifold.py`	first-generation covariance library
`code/ch0figs_en.py`	Chapter 0 figures (Life, ECA, light cone, reversibility, self-similarity)
`code/diagrams_en.py`	the five conceptual diagrams (island, dimensions, bulk/boundary, domain wall, RT geodesic)
`code/thermo.py`	full state enumeration for Life and ECA; coarse/fine entropy; Floquet quasi-energies
`code/chsh.py`, `chsh2.py`, `chsh3.py`	CHSH observables, Wick correlators, local frame optimisation, Ising chain
`code/maxwell.py`	lattice Bianchi identity, dispersion, $\mathbb{Z}_N$ charge
`code/massratio.py`	two-body problem on a lattice, the B term, mass-ratio scan
`code/wall3.py`	effective potential, relaxed kink, three settings
`code/sector2.py`	$\mathbb{Z}_2$ gauge two-species model, physical-sector projection, conserved-quantity search
`code/rigidity.py`	constrained Peierls model, Type I rigidity test
`code/ladder.py`	verification of $U_q(N)$, the Type II ladder, the rigid invariant
`figures/`	109 files (44 from the first generation, 42 from editions 2-9, 22 from the tenth edition, 1 from the twelfth). The archive carries **the complete set**
`code/`, tenth edition	The deformation algebra (causal set, continuum ADM flow, viscosity solution), the sparse link representation, the tilted light cone, and the radial solver for higher-order flux laws. All shipped with their calibrations in the `*_bundle.tar.gz` archives
`secondlaw_bundle.tar.gz`	The computations of Chapter 86 (`sl_run1` – `sl_run6`, logs, declarations, plotting), with calibrations
`code/bundles/`	Code, logs, declarations and result documents of each `*_bundle.tar.gz`. Included in the release from the thirteenth edition
`code/v11/`	The eleventh-edition check script (K403–K407) and drafts
`code/peer_review/`	The checks of Chapter 87 (`pr_check.py`), the referee report and its verification

Code and declarations are included. Of the 279 numbers from K100 up, 273 appear in the documents of the bundles and K401–K406 in the eleventh-edition drafts (`code/v11/`), which only confirms they are recorded. No script for the recalculation of K400–K402 could be found. In the thirteenth edition rerunning was confirmed only for K220–K223 and the checks of Chapter 87. If a number is not believable, run the calibration first.

The rule just keeps turning. But which way it turns is not decided.